Understanding the nucleon structure: basic formalism, modern tools and open questions

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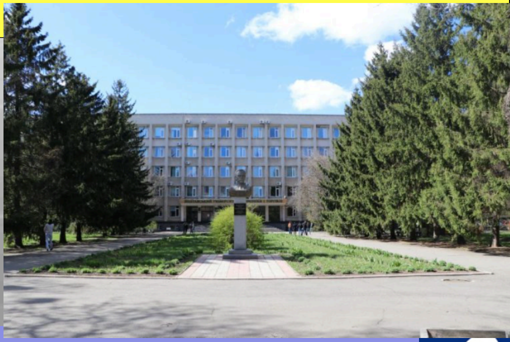

Outline

Part I

- Introduction
  - Motivation and scales
- Phenomenology
  - Elementary reactions
    - kinematics, crossing symmetry
    - Elastic scattering
    - Jlab and the GEp experiment
- Applications
    - Polarization
    - The proton radius problem

Part II

- Phenomenology
  - Elementary reactions
    - Annihilation reactions
    - The PANDA experiment
- Form factors in annihilation and scattering:
  understanding how matter is formed

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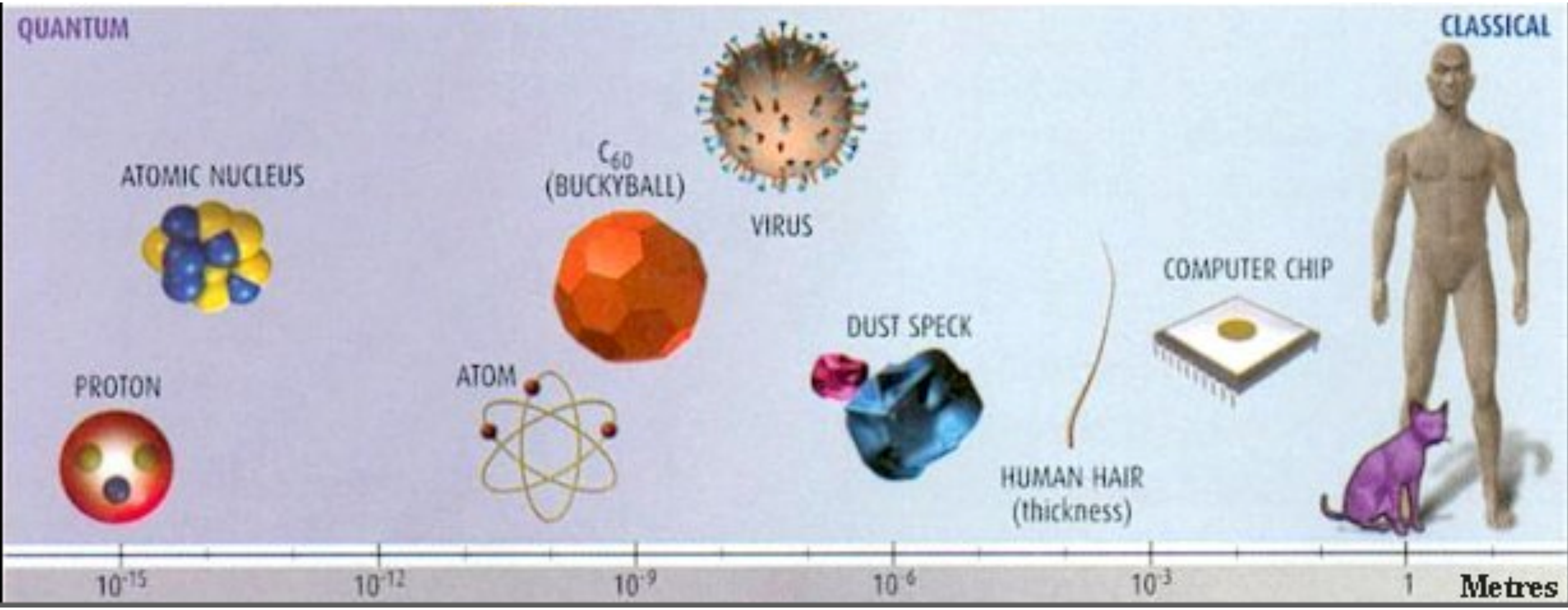
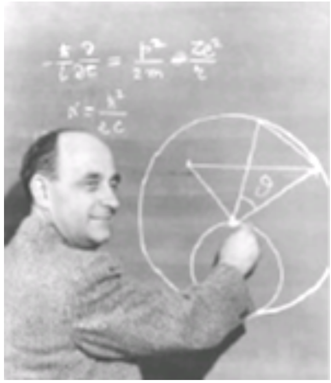
Scales

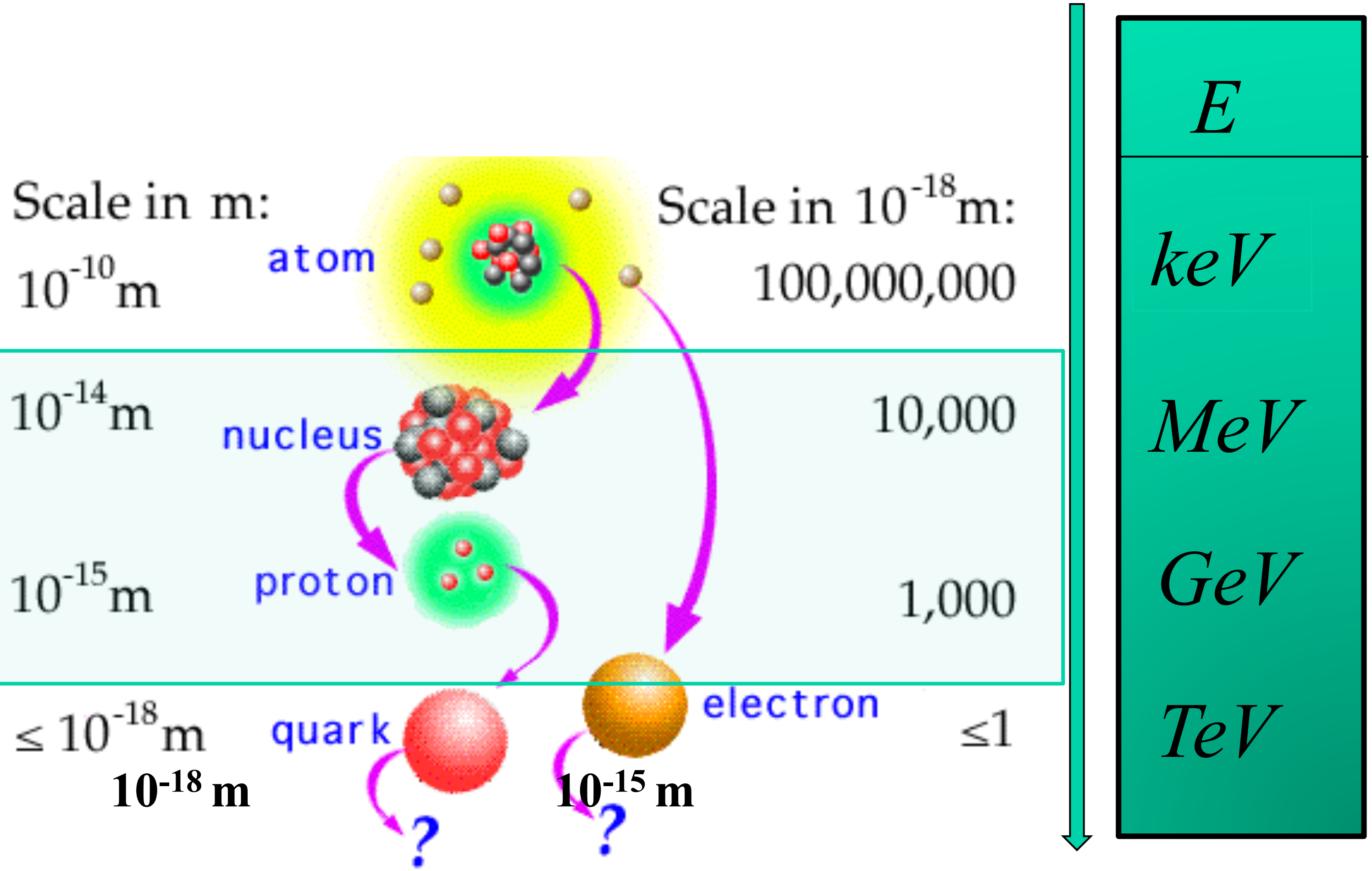


10^{-15} m 10^{-12} m 10^{-9} m 10^{-6} m 10^{-3} m 1 m

Particle, Nuclear, Hadronic physics

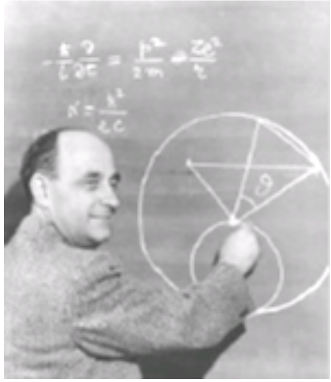


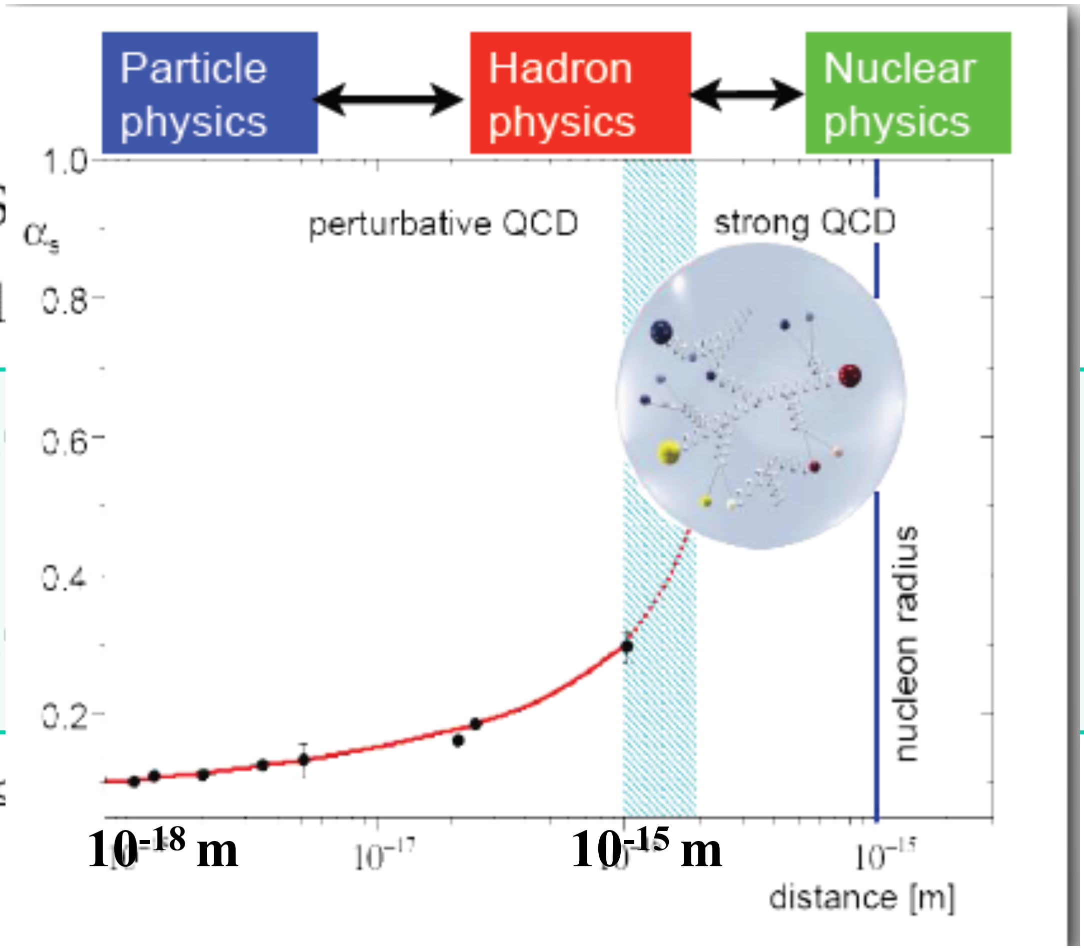
1 fm



- What is the microscope?
- What to look for?

Particle, Nuclear, Hadronic physics





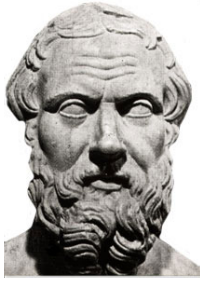
1 fm

E
keV
MeV
GeV
TeV

- *What is the microscope?*
- *What to look for?*

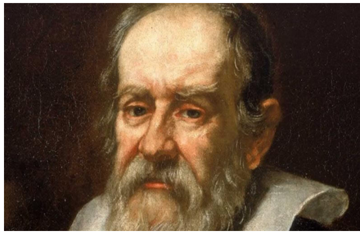
What are the costituents of matter?

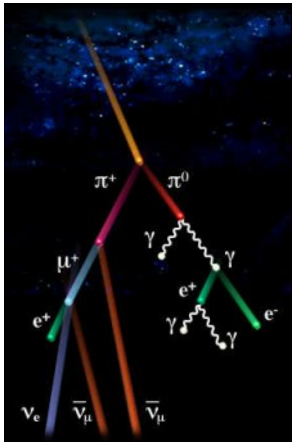
From the 4 elements: Air, Water, Earth, Fire.....



(470-360 BC) Democritus : atoms and vacuum
discretization, mechanicism

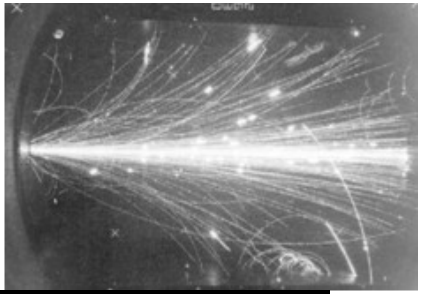
(1564-1642 AD) Galileo : scientific method
observation, hypothesis, experiment and formalize in mathematical form





XX century: Discovery of new particles:
electron, proton, neutrino, muon, mesons,baryons…

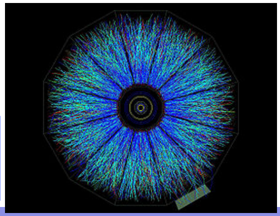
even too many…



Cloud chamber

Classification with quarks and leptons
(Standard Model)

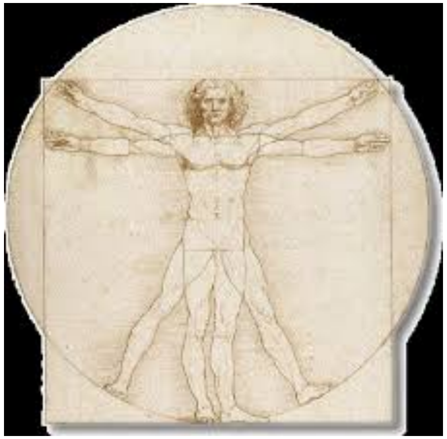
STAR at RHIC



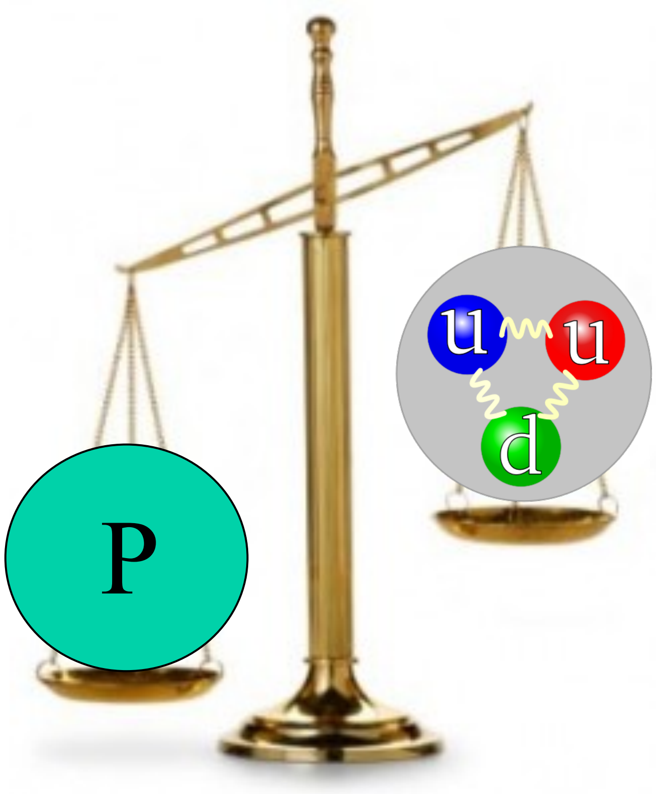
The proton

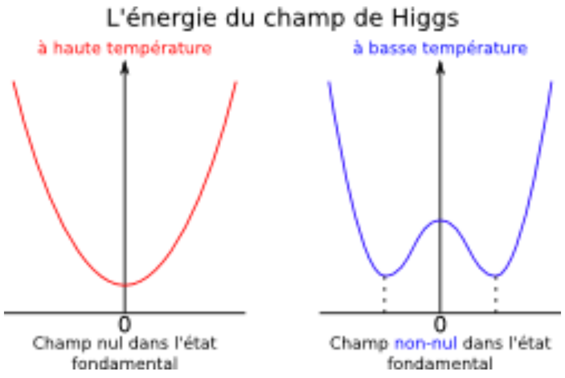
- Hadrons : the most part of visible matter
- Proton is the the most common particle in nature
- Its fundamental properties as
  - Mass
  - Size
  - Spin

  are still object of controversy



The proton MASS

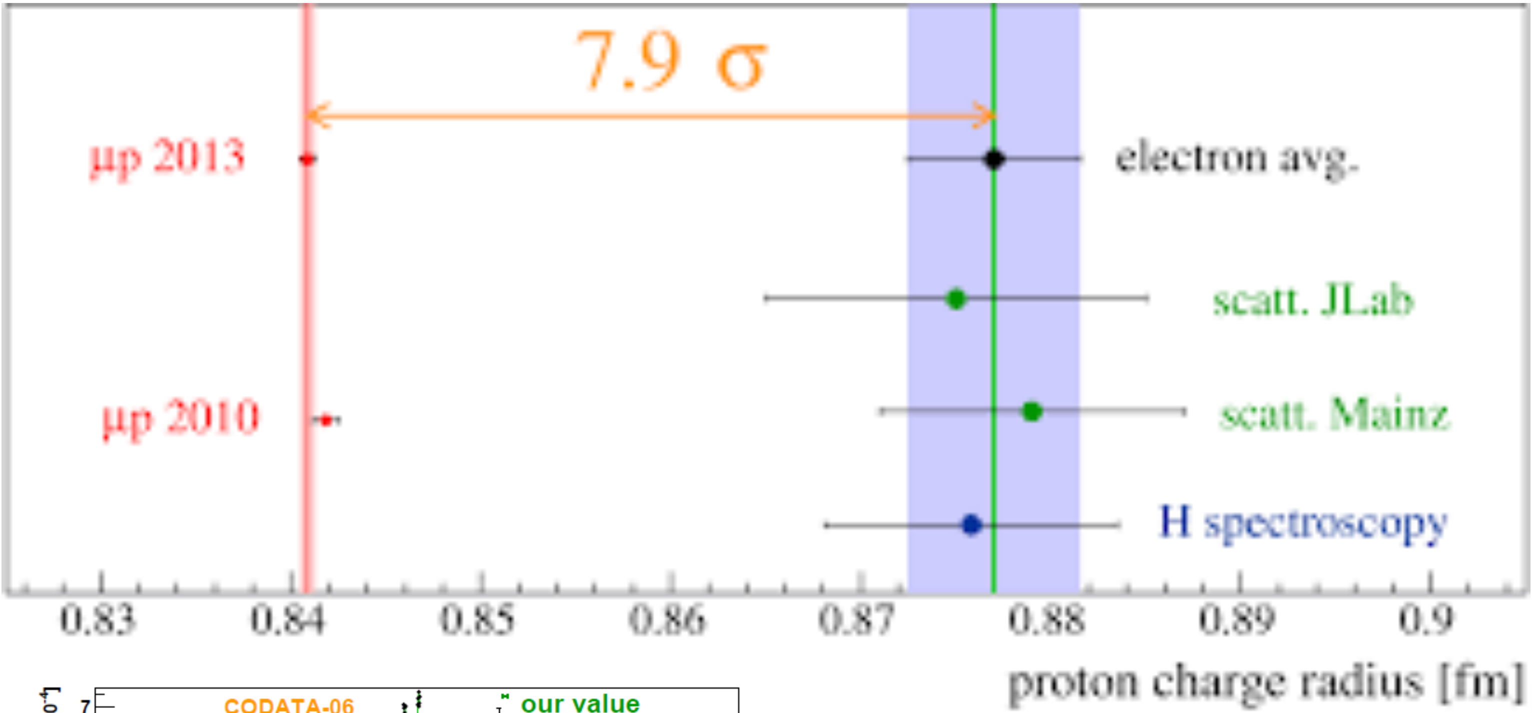


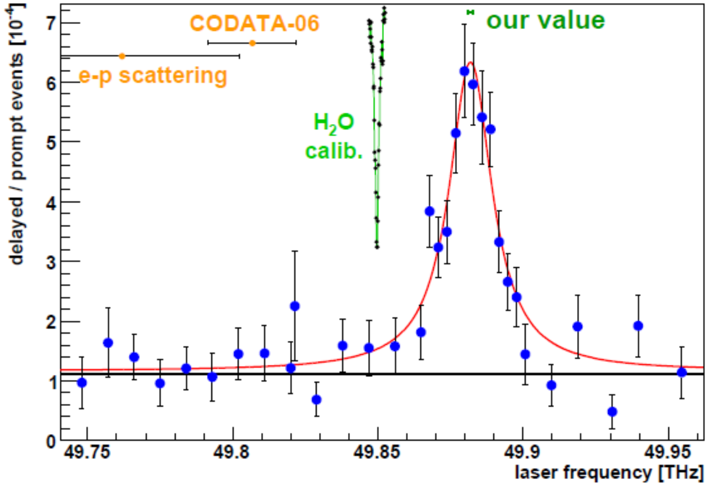


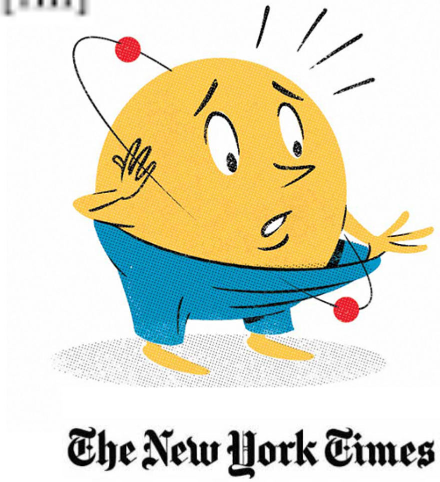
Mass
u-quark=1.5-4 MeV/c^2
d-quark=4-8 MeV/c^2

Mp=938,2720 MeV/c^2

The Proton Size (Radius)

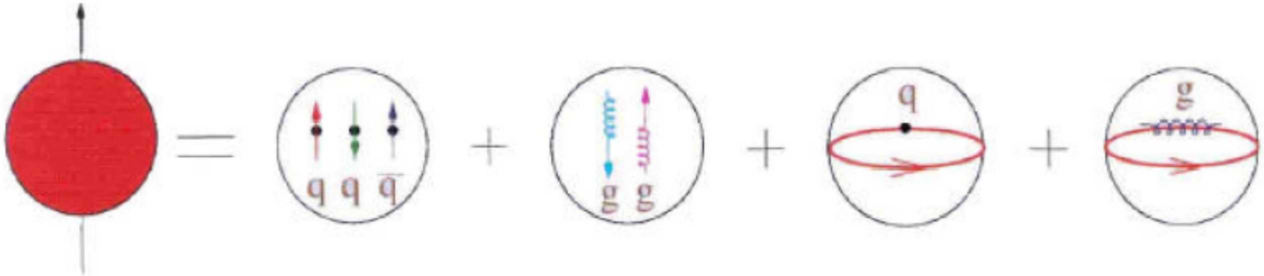






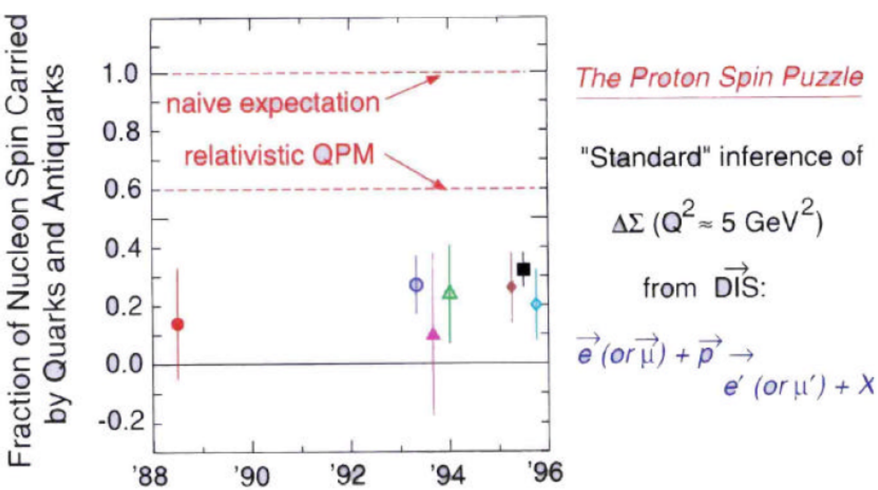
The proton electromagnetic structure (I)

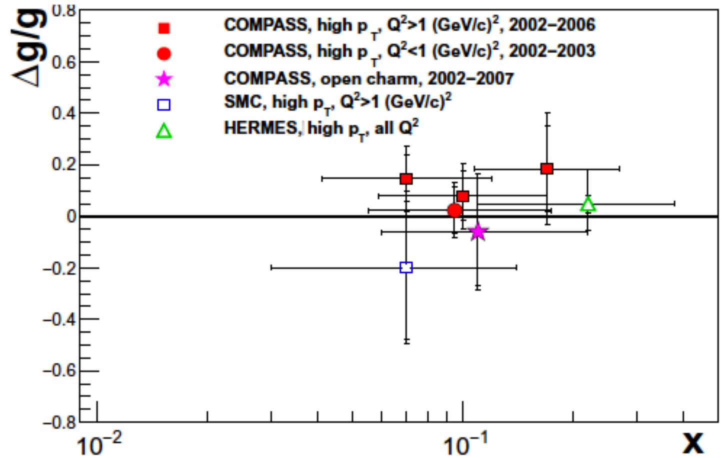
- Who carries the proton spin?



Proton Spin = Quark Spin + Gluon Spin + Quark Orbital + Gluon Orbital

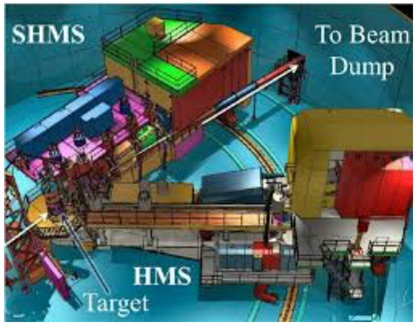
$$\langle S_{p_z} \rangle = \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + \langle L_{q_z} \rangle + \langle L_{G_z} \rangle$$



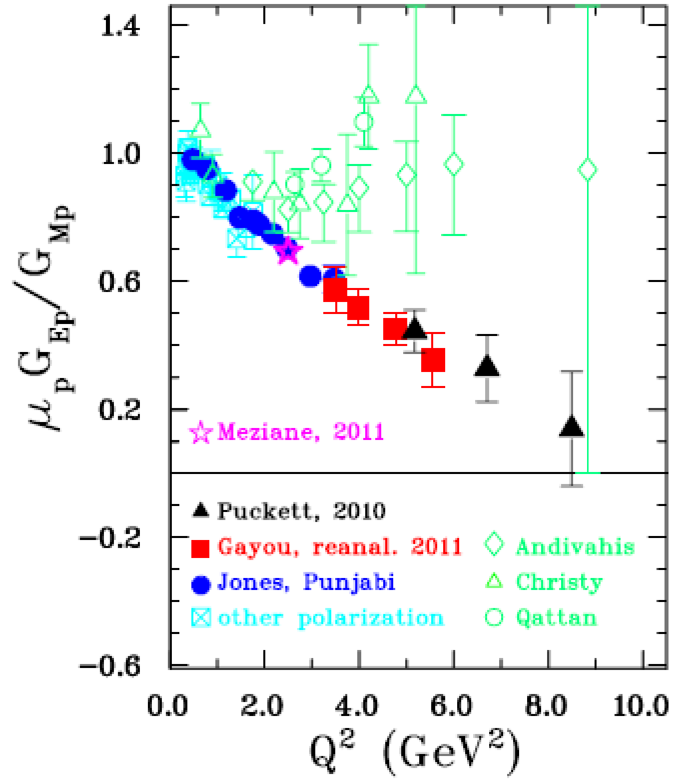


The proton electromagnetic structure (II)

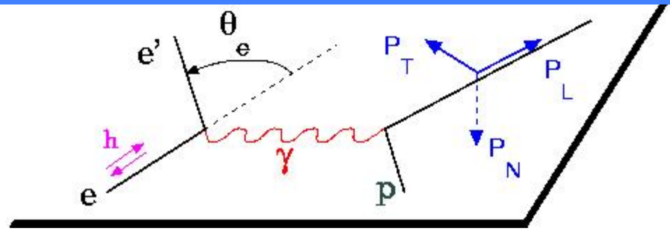
- How is the EM charge distributed?



Jefferson Lab

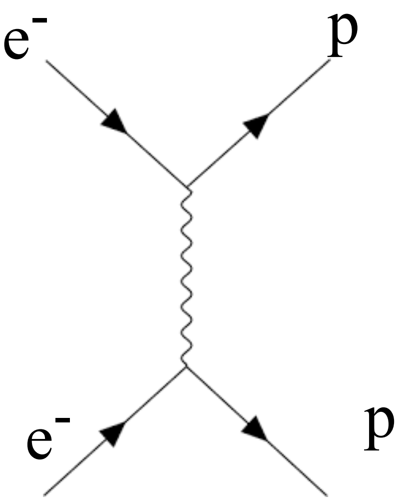


A.I. Akhiezer and M.P. Rekalo, Kharkov (1967)



Elementary Reactions

$e^-(k_1) + p(p_1) \longrightarrow e^-(k_2) + p(p_2)$

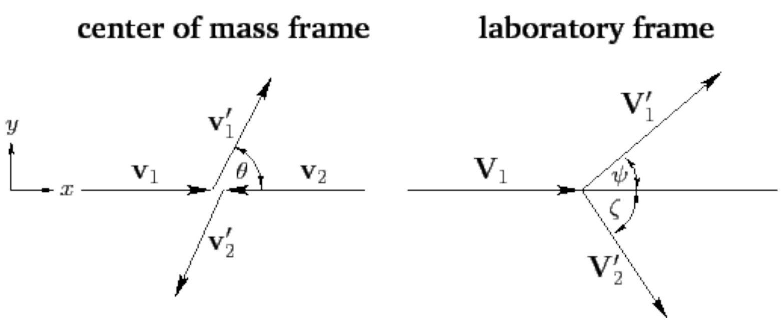


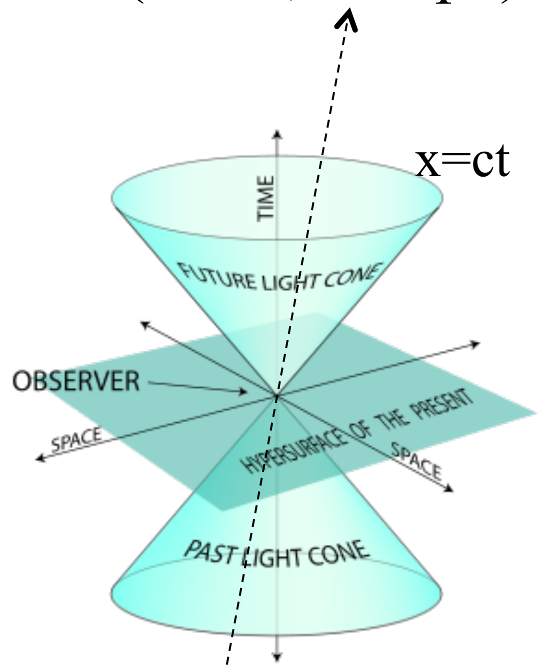
4-vector

$k_1 = (\varepsilon_1, \vec{k}_1)$

$k_1^2 = \varepsilon_1^{\,2} - \vec{k}_1^{\,2} = m^2$

$P = (t \text{ or } E, \vec{x} \text{ or } \vec{p})$





R.Feyman
(1918-1988)

4-momentum conservation

$k_1 + p_1 = k_2 + p_2$

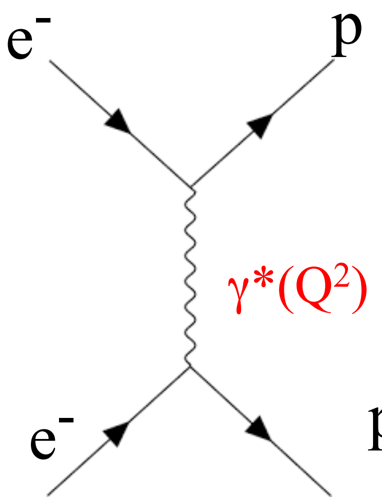
$q^2 = (k_1 - k_2)^2 = -4E_1E_2\sin^2\theta/2 < 0$ *SL*

$q^2 = (p_2 - p_1)^2 = 2mT = 2M(E_2 - M)$

Space-like
Time-like

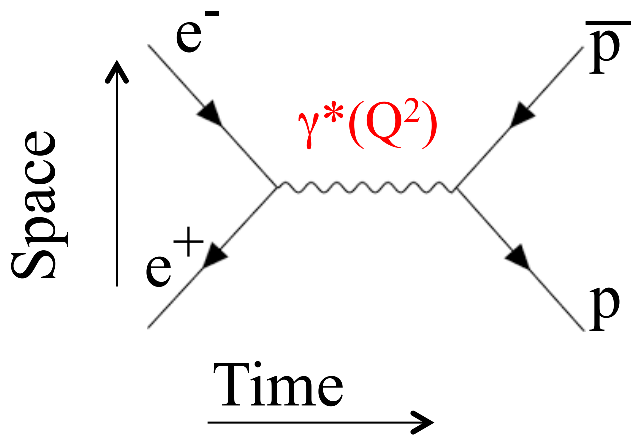
Elementary reactions

$e^-(k_1) + p(p_1) \longrightarrow e^-(k_2) + p(p_2)$	*Scattering*
$p(p_1) + \overline{p}(-p_2) \longrightarrow e^-(k_2) + e^+(-k_1)$	*Annihilation*
$e^-(k_1) + e^+(-k_2) \longrightarrow \overline{p}(-p_1) + p(p_2)$	



The interaction occurs through the exchange of a virtual photon of 'mass' Q^2

$Q^2=t=(k_1-k_2)^2$: *t-channel*
$Q^2=s=(k_1+p_1)^2$: *s-channel*

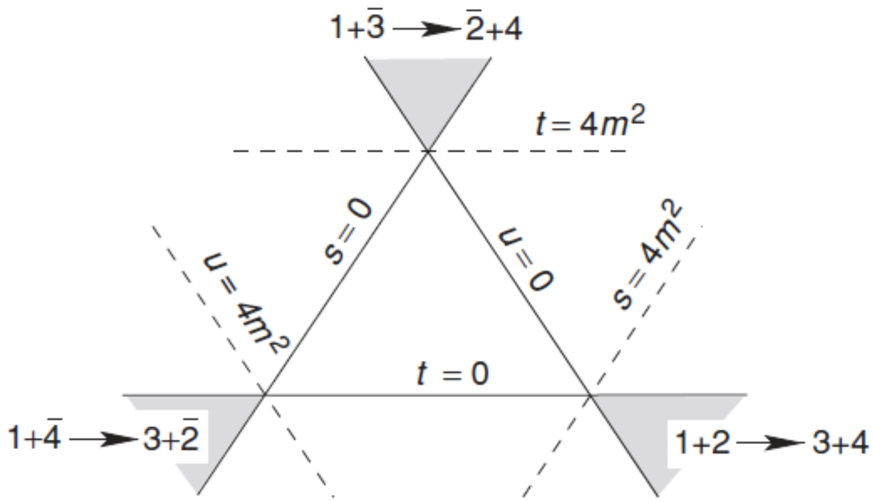


Crossed channels:

- described by the same amplitude *f(s,t)*
- a particle becomes antiparticle
- 4-momenta change sign
- scan different kinematical regions

Mandelstam Variables

$e^-(k_1) + p(p_1) \longrightarrow e^-(k_2) + p(p_2)$



Mandelstam variables: Lorentz invariants!

Total energy: $s=(k_1+p_1)^2$

Transferred momentum (1-3): $t=(k_1-k_2)^2$

Transferred momentum (1-4): $u=(k_1-p_2)^2$

$$s+t+u=\Sigma m_i^2$$

S.Mandelstam
(1928-2016)

Exercise

$\tilde{\theta}$: emission angle of the proton produced in annihilation (CMS system)

θ: scattered electron angle in ep elastic scattering (LAB system)

- Express as function of invariants
- Prove that:

$$cos^2\tilde{\theta} = 1 + \frac{st + (s-M^2)^2}{t(\frac{t}{4} - M^2)} \rightarrow 1 + \frac{ctg^2\frac{\theta}{2}}{1+\tau}$$

$\tau=Q^2/(4M^2)$

Cross section (Exp)

$$e^-(k_1) + p(p_1) \longrightarrow e^-(k_2) + p(p_2)$$

transition rate : $N_F = \sigma N_b N_T$

- The cross section can be understood as 'an effective area' over which the incident particle reacts

  Dimensions $[L^2]$: cm^2, or barn   1 barn$=10^{-28}$ m^2

- Luminosity $\mathcal{L}$: operative definition, useful for counting rate estimates

  $\mathcal{L} = N_b[s^{-1}]\ N_T\ [cm^{-2}]$, $dN_f/dt\ [s^{-1}] = \sigma\ \mathcal{L}$

.

Cross section (Th)

A relativistic definition of σ must
- imply relativistic kinematics ;
- have relativistic invariant form ;
- M= f (s, t, u) be expressed as a function of relativistic invariants

$$d\sigma = (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\ |\overline{M}|^2 d\mathcal{P} / \mathcal{I}$$

4 terms :

- The matrix element M, contains the dynamics of the reaction, calculated following a model ;
- The flux of colliding particles $\mathcal{I}$;
- The phase space for the final particles, $d\mathcal{P}$;
- Contains the conservation of four-momentum : $\delta^4(p_1+p_2-p_3-p_4)$ (product of four δ functions)

The incident flux

Two equivalent expressions

LAB system: $p_1 = (E_1, \vec{p}_1), \quad p_2 = (M_2, 0),$

$$\mathcal{I} = n_B n_T v_{rel},$$

Density → energy

$$n_B = 2E_1, \; n_T = 2M_2.$$

$$|\vec{v}_{rel}| = |\vec{v}_1 - \vec{v}_2| = \frac{|\vec{p}_1|}{E_1}$$

$$\mathcal{I} = 2E_1 2M_2 \frac{|\vec{p}_1|}{E_1} = 4M_2|\vec{p}_1|$$

$$\mathcal{I} = 4\sqrt{(p_1 \cdot p_2)^2 - M_1^2 M_2^2},$$

$$(p_1 \cdot p_2)^2 - M_1^2 M_2^2 = M_2^2 E_1^2 - M_1^2 M_2^2$$
$$= M_2^2 (E_1^2 - M_1^2) = M_2^2 |\boldsymbol{p}_1|^2$$

$$\mathcal{I} = 4M_2|\vec{p}_1|$$

The phase space

$$dP = \int \frac{d^4p\ \delta(p^2 - M^2)}{(2\pi)^3} \Theta(E),$$

QM definition of number of states in the unit volume (3dim)

Extracting the term that depends on energy:

$$d^4p\ \delta(p^2 - M^2) = \delta^3\vec{p} dE \delta(E^2 - \vec{p}^2 - M^2),$$

$$f(E) = E^2 - \vec{p}^2 - M^2,$$

Properties of δ function, x_i are the root of $f(x)$:

$$\int \delta[f(x)]dx = \sum \frac{1}{|f'(x_i)|},$$

$$\int dE \delta(E^2 - \vec{p}^2 - M^2)\Theta(E) = \frac{1}{2E}.$$

Phase space for 1+2→3+4:

$$dP = \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}.$$

The cross section

$$\sigma = \frac{(2\pi)^4}{\mathcal{I}} \int |\mathcal{M}|^2 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}.$$

$$\int \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) d^3\vec{p}_4 = 1$$ Properties of δ function

$$\mathcal{J} = \delta(E_1 + E_2 - E_3 - E_4) \frac{d^3\vec{p}_3}{4E_3E_4} = \delta(W - E_3 - E_4) \frac{|\vec{p}|^2 d\Omega dp}{4E_3E_4},$$

$W = E_1 + E_2$
$= E_3 + E_4$

In CMS:

$$E_3^2 = M_3^2 + |\vec{p}|^2, \ E_4^2 = M_4^2 + |\vec{p}|^2 \to E_3 dE_3 = E_4 dE_4 = |\vec{p}| dp.$$

$$\frac{d}{dE_3}(W - E_3 - E_4) = -1 - \frac{dE_4}{dE_3} = -1 - \frac{E_3}{E_4} = -\frac{W}{E_4}$$

$$\mathcal{J} = \int \delta(W - E_3 - E_4) \frac{dE_3 |\vec{p}| d\Omega}{4E_4} = \frac{|\vec{p}| d\Omega}{4E_4} \frac{1}{\left| \frac{d}{dE_3}(W - E_3 - E_4) \right|} = \frac{|\vec{p}| d\Omega}{4W}.$$

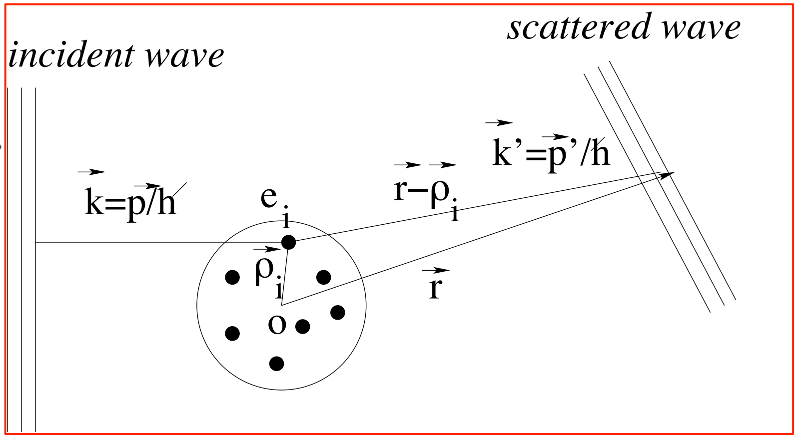
$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2} |\vec{p}|}{64\pi^2 W^2 |\vec{k}|},$$ CMS

$$\frac{d\sigma}{d\Omega_e} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2} \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \frac{1}{M^2}$$ LAB

Proton radius & charge form factor

The amplitude of the scattered wave in the point defined by $\vec{r}$ is :

$$A_i = f e_i e^{i\vec{k}\cdot\vec{\rho}_i} e^{i\vec{k}'\cdot(\vec{r}-\vec{\rho}_i)} = f e^{i\vec{k}\cdot\vec{r}} e_i e^{iq\cdot\vec{\rho}_i}$$



The total scattered amplitude on the nucleus is the sum of the amplitudes on the individual charges :

$$A = \sum_i A_i = f e^{i\vec{k}\cdot\vec{r}} \sum_i e_i e^{iq\cdot\vec{\rho}_i}, \quad f \simeq Z_a e.$$

$\vec{\rho}_i$ Position operator of the internal motion in the target

➔ Mean value in the ground state of the target

Definition of form factor

$$F(\vec{q}) = \frac{1}{Z_b e} < i| \sum_i e_i e^{i\vec{q}\cdot\vec{\rho}_i} |i>$$

Definition of cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{pl} |F(\vec{q})|^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{pl} = (Z_b e)^2 |f|^2 \propto (Z_a Z_b e^2)^2$$

Inverting the definition:
Mean value of the charge density

$$\rho(\vec{x}) =< \Psi_i|\hat{\rho}(\vec{x})|\Psi_i >= \frac{Z_2 e}{(2\pi)^3} \int d^3q F(\vec{q}) e^{-i\vec{q}\cdot\vec{x}}.$$

In practice one assumes a charge density, Fourier transform, fits the function on experimental data

Root mean square radius <r^2>

Taylor expansion:

$$
\begin{aligned}
F(\vec{q}) &= \frac{1}{Z_b e}\int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x}) \\
&= \frac{1}{Z_b e}\int d^3\vec{x}\left[1 + i\vec{q}\cdot\vec{x} - \frac{1}{2}(\vec{q}\cdot\vec{x})^2 + \ldots\right]\rho(\vec{x}) = \\
&= \frac{1}{Z_b e}\int_0^\infty x^2 dx \int_0^{2\pi} d\phi \\
&\int_{-1}^{1} d\cos\theta\left[1 + iqx\cos\theta - \frac{1}{2}q^2x^2\cos^2\theta\right]\rho(\vec{x})
\end{aligned}
$$

1. $\int_\Omega d^3\vec{x}\rho(\vec{x}) = Z_b e.$
2. $\int_{-1}^{1}\cos\theta\, d\cos\theta = 0$

wave function is even with respect to space parity

Assuming spherical symmetry

$$< r_c^2 >= \frac{\int_0^\infty x^4\rho(x)dx}{\int_0^\infty x^2\rho(x)dx}.$$

$$F(q) \sim 1 - \frac{1}{6}q^2 < r_c^2 > + O(q^2),$$

$$\langle r^2_{E/M}\rangle = -\frac{6\hbar^2}{G_{E/M}(0)}\frac{dG_{E/M}(Q^2)}{dQ^2}\Bigg|_{Q^2=0}.$$

RMS is the limit of the form factor derivative for $Q^2 \to 0$

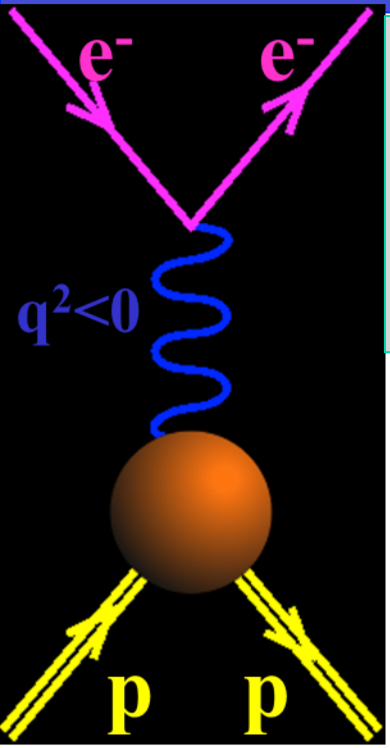
Root mean square radius

$$F(q) = \frac{\int_\Omega d^3\vec{x} e^{i\vec{q}\cdot\vec{x}} \rho(\vec{x})}{\int_\Omega d^3\vec{x} \rho(\vec{x})}.$$

In *non-relativistic approach*
(and also in relativistic but in *Breit frame*)
FFs are Fourier transform of the density

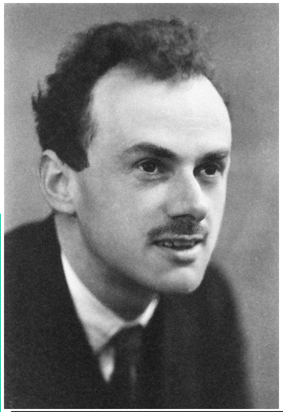
density $\rho(r)$	Form factor $F(q^2)$	r.m.s. $< r_c^2 >$	comments
δ	1	0	pointlike
e^{-ar}	$\frac{a^4}{(q^2+a^2)^2}$	$\frac{12}{a^2}$	dipole
$\frac{e^{-ar}}{r}$	$\frac{a^2}{q^2+a^2}$	$\frac{6}{a^2}$	monopole
$\frac{e^{-ar^2}}{r^2}$	$e^{-q^2/(4a^2)}$	$\frac{1}{2a}$	gaussian
ρ_0 for $x \leq R$ 0 for $r \geq R$	$\frac{3(\sin X - X\cos X)}{X^3}$ $X = qR$	$\frac{3}{5}R^2$	square well

Elementary reactions



- Disentangle reaction mechanism and internal structure
  - The electron vertex is known
  - The virtual photon propagator $1/q^2$
  - The proton structure is contained in the vertex γ^*pp
    Pauli and Dirac form factors

- The EM Lepton current $\ell_\mu = \overline{u}(k_2)\gamma_\mu u(k_1)$



P.A.M. Dirac
(1902-1984)

- The EM Hadron current

$$\mathcal{J}_\mu = \overline{u}(p_2)\left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p_1),$$
$$\sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2}.$$

W. Pauli
(1902-1984)

- A simpler form:

$$\mathcal{J}_\mu = \overline{u}(p_2)[(F_1+F_2)\gamma_\mu - \frac{(p_1+p_2)_\mu}{2m}F_2]u(p_1).$$

$$\mathcal{M} = \frac{e^2}{q^2}\ell_\mu\mathcal{J}_\mu = \frac{e^2}{q^2}\ell\cdot\mathcal{J}$$ Matrix element

The Matrix Element Squared

- The matrix element squared

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 \overline{|\ell \cdot \mathcal{J}|^2} = \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu} W_{\mu\nu},$$

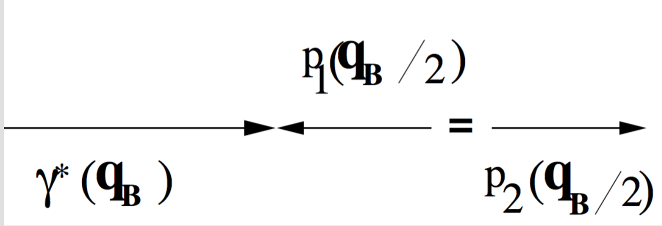
- The leptonic tensor $L_{\mu\nu} = \overline{\ell_\mu \ell_\nu^*}$
- The hadronic tensor $W_{\mu\nu} = \overline{\mathcal{J}_\mu \mathcal{J}_\nu^*}$

G. Breit
(1899-1981)

Relativistic invariants: can be calculated in any system:

- The experiment is in Lab system
- The theorie: any system

	Lab	CMS	Breit
q	$(\omega, \mathbf{q})$	$(\widetilde{\omega}, \widetilde{\mathbf{q}})$	$(\omega_B = 0, \mathbf{q_B})$
k_1	$(\epsilon_1, \mathbf{k_1})$	$(\widetilde{\epsilon_1}, \widetilde{\mathbf{k_1}})$	$(\epsilon_{1B}, \mathbf{k_{1B}})$
p_1	$(M_p, 0)$	$(\widetilde{E_1}, -\widetilde{\mathbf{k_1}})$	$(E_{1B}, \mathbf{p_{1B}})$
k_2	$(\epsilon_2, \mathbf{k_2})$	$(\widetilde{\epsilon_2}, \widetilde{\mathbf{k_2}})$	$(\epsilon_{2B}, \mathbf{k_{2B}})$
p_2	$(E_2, \mathbf{p_2})$	$(\widetilde{E_2}, -\widetilde{\mathbf{k_2}})$	$(E_{2B}, -\mathbf{p_{1B}})$



Breit system

(unpolarized) Lepton tensor

$\overline{u} = u^\dagger \gamma_0,\ \overline{u}^\dagger = (u^\dagger \gamma_0)^\dagger = \gamma_0^\dagger u = \gamma_0 u,\ \gamma_0 \gamma_0 = 1,\ \ \gamma_0^\dagger = \gamma_0,$

$$\begin{aligned}
L_{\mu\nu} &= \overline{\ell_\mu \ell_\nu^*} = \overline{\overline{u}(k_2)\gamma_\mu u(k_1)\left[\overline{u}(k_2)\gamma_\nu u(k_1)\right]^*}. \\
&= \overline{\overline{u}(k_2)\gamma_\mu u(k_1) u^\dagger(k_1)\gamma_\nu^\dagger \overline{u}(k_2)} \\
&= \overline{\overline{u}(k_2)\gamma_\mu u(k_1) u^\dagger(k_1)\gamma_0\gamma_0\gamma_\nu^\dagger \gamma_0 u(k_2)} \\
&= \overline{\overline{u}(k_2)\gamma_\mu u(k_1) u^\dagger(k_1)\gamma_\nu^\dagger \gamma_0 u(k_2)} = \frac{1}{2} Tr \gamma_\mu \rho_1 \gamma_\nu \rho_2.
\end{aligned}$$

$\rho = \overline{u(k) u^\dagger(k)} = \hat{k} + m_e,$
$Tr\gamma_a\gamma_b = 4g_{ab}$ ($g_{ab} = 1$, for $a, b = 0$ $g_{ab} = -1$, for $a, b = x, y, z$;
$Tr\gamma_a\gamma_b\gamma_c\gamma_d = 4(g_{ab}g_{cd} + g_{bc}g_{da} - g_{ac}g_{bd})$

$$L_{\mu\nu} = 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} + 2g_{\mu\nu}(m^2 - k_1 \cdot k_2)$$

Using that $k_1 = q + k_2$; $q^2 = 2(m^2 - k_1 \cdot k_2)$ we find :

$$\boxed{L_{\mu\nu} = 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} + g_{\mu\nu}q^2.}$$

symmetric tensor

The (unpolarized) Hadronic Tensor

R.G. Sachs
(1916-1999)

In the Breit system, the hadronic tensor has a simple physical meaning in terms of the Sachs FFs, linear combinations of Dirac and Pauli FFs.

$G_M = F_1 + F_2$	$F_1 = \dfrac{G_E + \tau G_M}{1+\tau}$
$G_E = F_1 - \tau F_2$	$F_2 = \dfrac{G_M - G_E}{1+\tau}$

Let us write $\mathcal{J}_\mu = \chi_2^\dagger F_\mu \chi_1$

$$F_\mu = 2mG_E, \ \mu = 0$$
$$F_\mu = i\vec{\sigma} \times \mathbf{q_B} G_M, \ \mu = x, y, z.$$

$$F_\mu = \begin{cases} 2mG_E & ,\mu = 0 \\ i\sqrt{-q^2}G_M\sigma_y & ,\mu = x \\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu = y \\ 0 & ,\mu = z \end{cases}$$

The hadronic tensor $W_{\mu\nu}$:

$$W_{\mu\nu} = \overline{(\chi_2^\dagger F_\mu \chi_1)(\chi_1^\dagger F_\nu^\dagger \chi_2)} = \frac{1}{2} Tr F_\mu \rho_1 F_\nu^\dagger \rho_2;$$

In case of unpolarized particles $\rho = \mathcal{I}$, and

$$W_{\mu\nu} = \frac{1}{2} Tr F_\mu F_\nu^\dagger.$$

Overline:
average over the spins of the *initial* state
sum over the spins of the *final* state
From the *Fermi Golden Rule* that gives the transition probability between two states

The Rosenbluth Formula

- The terms containing the product $G_E G_M$ vanish $Tr\vec{\sigma}\cdot\mathbf{A}=0$,
  the unpolarized cross section does not contain interference terms

In the Lab system :

$$\overline{|\mathcal{M}|^2}=\left(\frac{e^2}{q^2}\right)^2 4m^2(-q^2)\left[2\tau G_M^2+\frac{\cot^2\frac{\theta_e}{2}}{1+\tau}(G_E^2+\tau G_M^2)\right].$$

The differential cross section

$$\frac{d\sigma}{d\Omega_e}=\frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\left[2\tau G_M^2+\frac{\cot^2\frac{\theta_e}{2}}{1+\tau}\left(G_E^2+\tau G_M^2\right)\right],$$

where $\alpha=e^2/4\pi\simeq 1/137$ is the fine structure constant.

M. Rosenbluth (1927-2003)

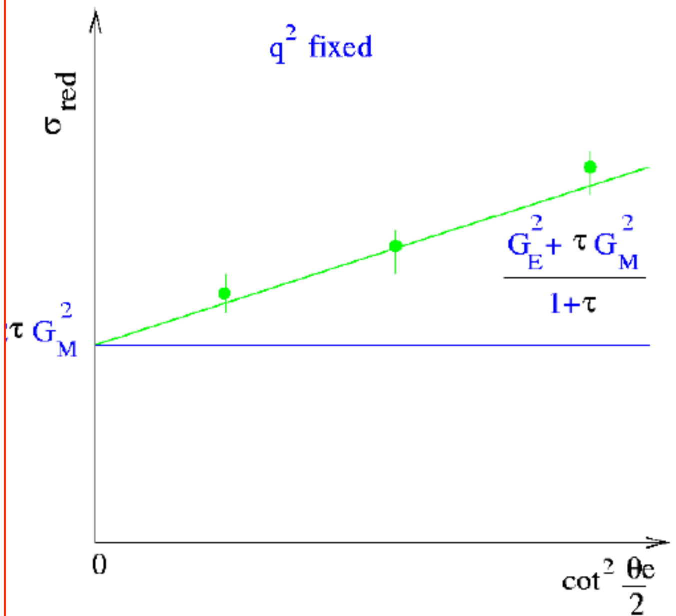
Exercice

Derive the relation between the electron scattering angle in the Breit and Laboratory system

$$\cot^2\frac{\theta_B}{2}=\frac{\cot^2\theta_e/2}{1+\tau}.$$

The Rosenbluth separation

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\left[2\tau G_M^2 + \frac{\cot^2\frac{\theta_e}{2}}{1+\tau}\left(G_E^2+\tau G_M^2\right)\right],$$

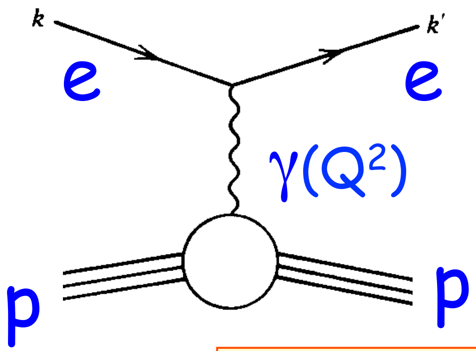


Linearity of the *reduced* cross section as a function of $\cot^2\frac{\theta_e}{2}$ $\rightarrow$ individual determination of G_E and G_M :

$$\sigma_{red} = \frac{\frac{d\sigma}{d\Omega_e}}{\frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2}$$

- The backward eN-scattering ($\theta_e=\pi$, $\cot^2\frac{\theta_e}{2}=0$) is determined by the magnetic FF only,
- The slope of σ_{red} is sensitive to G_E^2

Summary: ep-elastic scattering



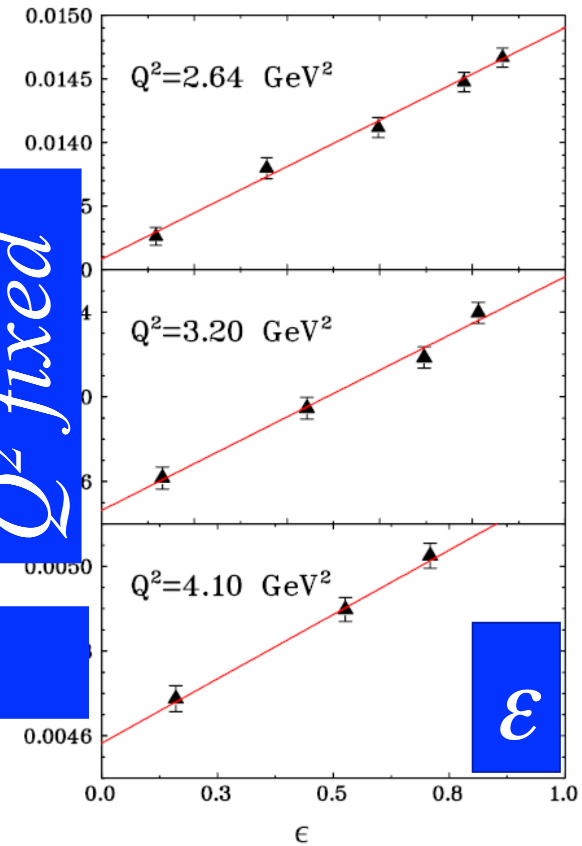
$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{Mott}\frac{1}{(1+\tau)}\left(G_E^2(Q^2)+\frac{\tau}{\varepsilon}G_M^2(Q^2)\right)$$

1950

$$\varepsilon=\left[1+2(1+\tau)\tan^2\left(\frac{\theta_e}{2}\right)\right]^{-1},\tau=\frac{Q^2}{4M^2}$$

$$\sigma_R=\varepsilon G_E^2+\tau G_M^2$$

Q^2 fixed



ε

Linearity of the reduced cross section

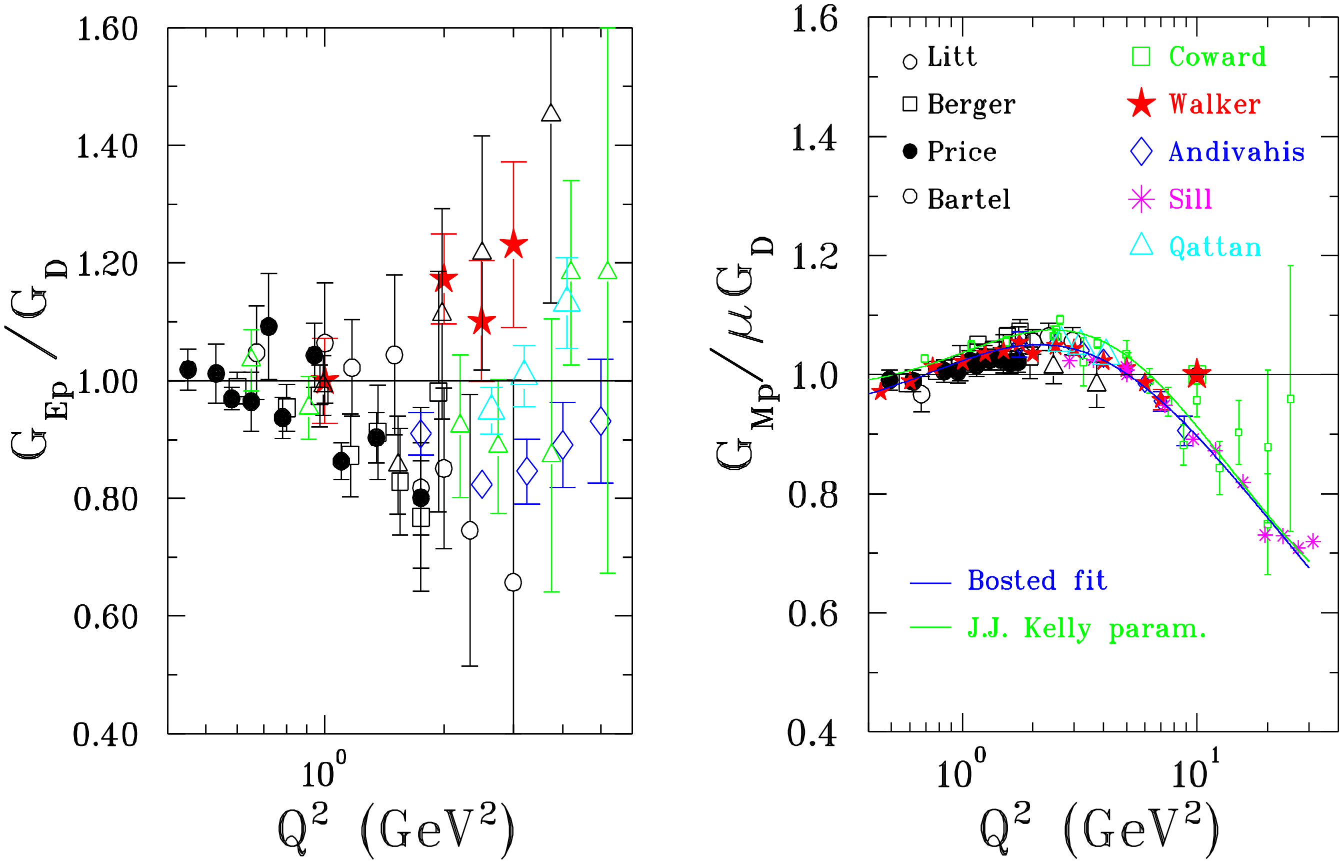
→ $\tan^2\theta_e$ dependence

→ Holds for 1γ exchange only

PRL 94, 142301 (2005)

Proton Form Factors...in last century

$G_M/\mu = G_E = G_D = (1+Q^2/0.71\ \mathrm{GeV}^2)^{-2}$:dipole aproximation

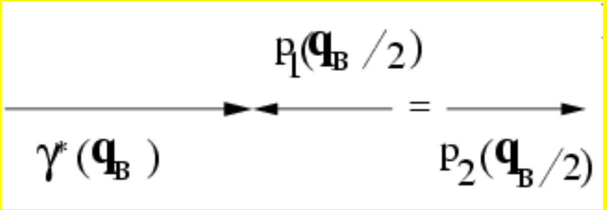


Rosenbluth separation/ Polarization observables

Dipole Approximation

$$G_D=(1+Q^2/\, 0.71\ GeV^2)^{-2}$$

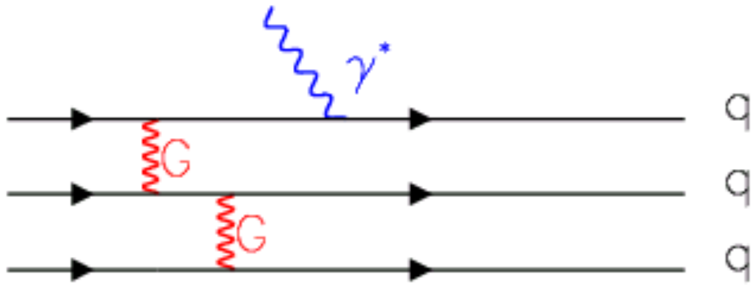
- Classical approach
  - Nucleon FF (in non relativistic approximation or in the Breit system) are Fourier transform of the charge or magnetic distribution.



Breit system

- The dipole approximation corresponds to **exponential density distribution.**
  - $\rho = \rho_0 \exp(-r/r_0)$,
  - $r_0^2 = (0.24\ fm)^2$, $<r^2> \sim (0.81\ fm)^2 \leftrightarrow m^2_D = 0.71\ GeV^2$

Dipole Approximation and pQCD



Dimensional scaling

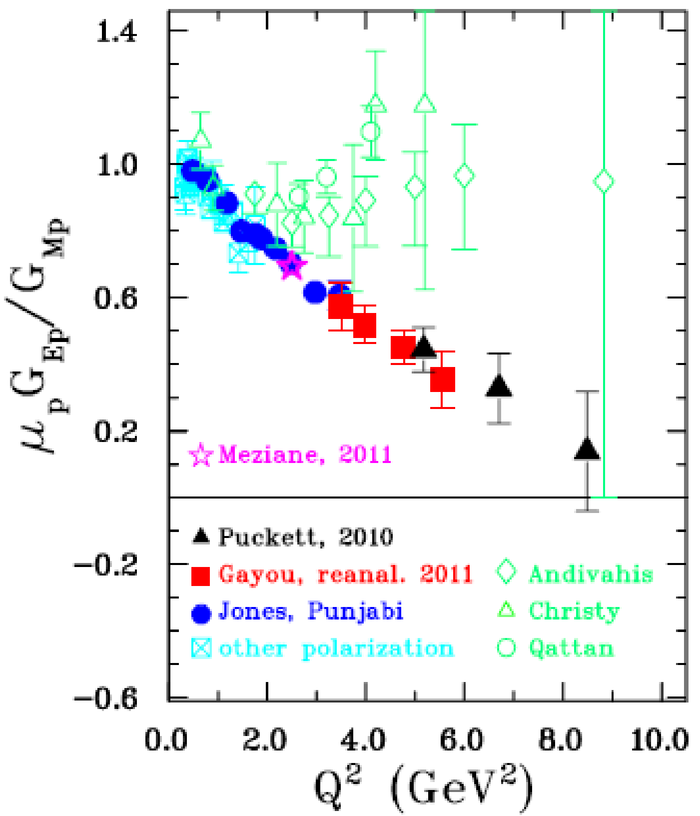
- $F_n(Q^2) = C_n\,[1/(1+Q^2/m_n)^{n-1}]$,
  - $m_n = n\beta^2$, <quark momentum squared>
  - n is the number of constituent quarks
- Setting $\beta^2 = (0.471 \pm .010)$ GeV^2 *(fitting pion data)*
  - pion: $F_\pi(Q^2) = C_\pi\,[1/(1+Q^2/0.471\ \mathrm{GeV}^2)^1]$,
  - nucleon: $F_N(Q^2) = C_N\,[1/(1+Q^2/0.71\ \mathrm{GeV}^2)^2]$,
  - deuteron: $F_d(Q^2) = C_d\,[1/(1+Q^2/1.41\,\mathrm{GeV}^2)^5]$

V. A. Matveev, R. M. Muradian, and A. N. Tavkhelidze (1973), Brodsky and Farrar (1973), Politzer (1974), Chernyak & Zhitnisky (1984), Efremov & Radyuskin (1980)…

Dipole Approximation

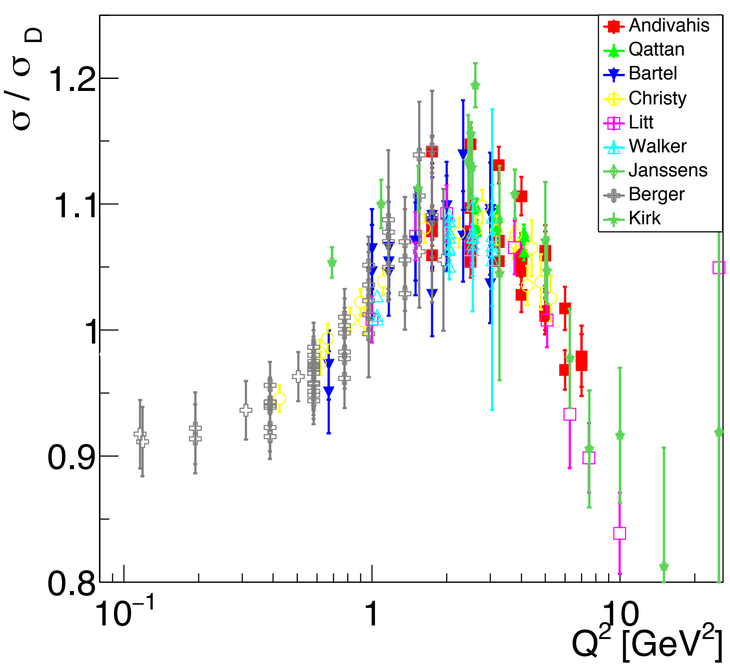
Does not hold in SL region:

neither for GE …nor for GM



The GEP collaboration,
A.J.R. Puckett et al, PRC96(2017)055203

R. Taylor, SLAC, 1967
S. Pacetti, E.T-G, PRC94, 055202 (2016)



…and in TL ?

SOVIET PHYSICS – DOKLADY VOL. 13, NO. 6 DECEMBER, 1968

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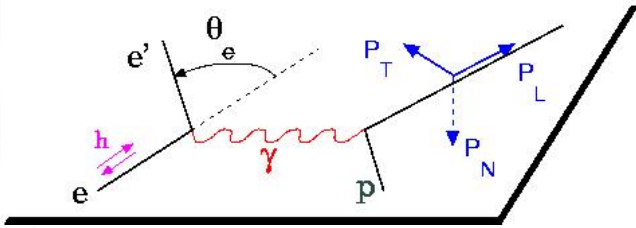
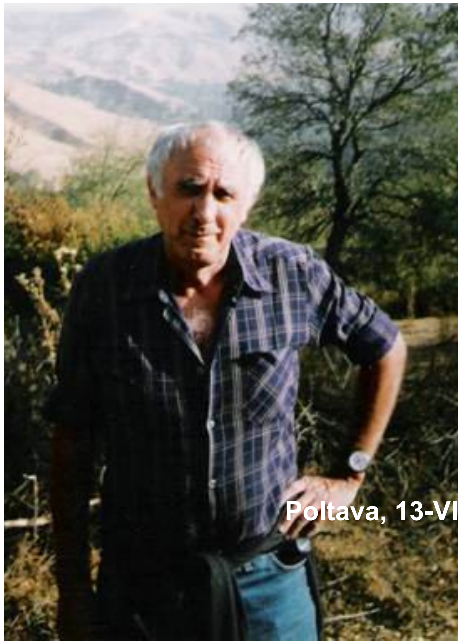


POLARIZATION PHENOMENA IN ELECTRON SCATTERING BY PROTONS IN THE HIGH-ENERGY REGION

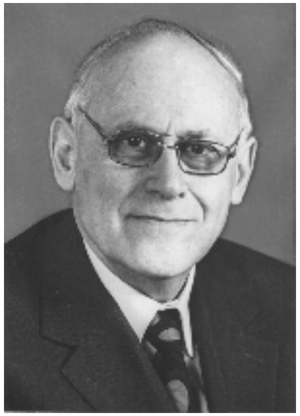
Academician A. I. Akhiezer* and M. P. Rekalo

Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR
Translated from Doklady Akademii Nauk SSSR, Vol. 180, No. 5, pp. 1081–1083, June, 1968
Original article submitted February 26, 1967

M.P. Rekalo
(1938-2004)



$$s_2 \frac{d\sigma}{d\Omega_R} = 4p_2 \frac{(s\cdot q)}{1+\tau}\Gamma(\theta,\ \varepsilon_1)\left[\tau G_M (G_M + G_E) - \frac{1}{4\varepsilon_1} G_M (G_E - \tau G_M)\right],$$



A.I. Akhiezer
(1911-2000)

The polarization induces a term
in the cross section proportional to $G_E\, G_M$
Polarized beam and target or polarized beam and recoil proton polarization

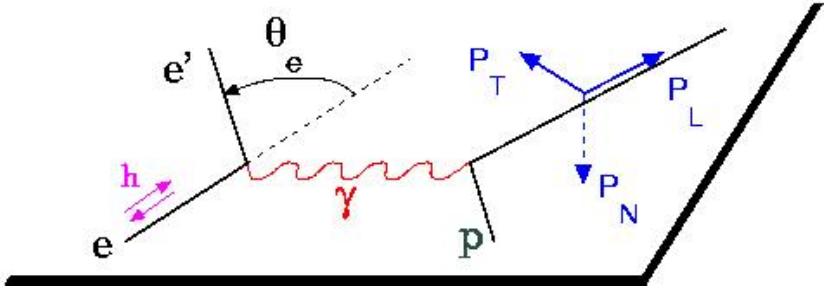
Polarized Hadron tensor

In general the hadronic tensor $W_{\mu\nu}$, for *ep* elastic scattering, contains four terms, related to the 4 possibilities of polarizing the initial and final protons :

$$W_{\mu\nu} = W^{(0)}_{\mu\nu} + W_{\mu\nu}(\mathbf{P_1}) + W_{\mu\nu}(\mathbf{P_2}) + W_{\mu\nu}(\mathbf{P_1}, \mathbf{P_2}),$$

$\mathbf{P_1}$ ($\mathbf{P_2}$) is the polarization vector of the initial (final) proton. The 2×2 density matrix for a nucleon with polarization $\mathbf{P}$:

$$\rho = \frac{1}{2}(1 + \vec{\sigma} \cdot \mathbf{P})$$



Polarized final proton ($\mathbf{P} = \mathbf{P_2}$) :

$$W_{\mu\nu}(\mathbf{P}) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \vec{\sigma} \cdot \mathbf{P}$$

For **longitudinally polarized electrons** on **unpolarized target**, only $\mathbf{P}_x \neq 0$ and $\mathbf{P}_z \neq 0$.

Polarized Hadron tensor Px

$$W_{\mu\nu}(P_x) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \sigma_x$$

F_μ	$F_\nu^\dagger$	$F_\nu^\dagger \sigma_x$	ν
$2mG_E$	$2mG_E$	$2mG_E\sigma_x$	0
$i\sqrt{-q^2}G_M\sigma_y$	$-i\sqrt{-q^2}G_M\sigma_y$	$-\sqrt{-q^2}G_M\sigma_z$	x
$-i\sqrt{-q^2}G_M\sigma_x$	$i\sqrt{-q^2}G_M\sigma_x$	i $\sqrt{-q^2}G_M$	y
0	0	0	z

$(\sigma_i\sigma_j = i\sigma_k, \sigma_y\sigma_x = -i\sigma_z)$ The nonzero components of $W_{\mu\nu}(P_x)$ are :

$$\begin{aligned} W_{0y}(P_x) &= iq^2 2mG_E G_M, \\ W_{y0}(P_x) &= -iq^2 G_E G_M, \end{aligned}$$

Polarized Hadron tensor Pz

The polarized tensor

$$W_{\mu\nu}(P_z) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \sigma_z .$$

F_μ	$F_\nu^\dagger$	$F_\nu^\dagger \sigma_z$	ν
$2mG_E$	$2mG_E$	$2mG_E\sigma_z$	0
$i\sqrt{-q^2}G_M\sigma_y$	$-i\sqrt{-q^2}G_M\sigma_y$	$-\sqrt{-q^2}G_M\sigma_x$	x
$-i\sqrt{-q^2}G_M\sigma_x$	$i\sqrt{-q^2}G_M\sigma_x$	$\sqrt{-q^2}G_M\sigma_y$	y
0	0	0	z

- $W_{0\nu}(P_z) = W_{\nu 0}(P_z) = 0$, for any ν,
- No interference term $G_E G_M$.
- The nonzero components of $W_{\mu\nu}(P_z)$ are :

$$W_{xy}(P_z) = -iq^2 G_M^2,$$
$$W_{yx}(P_z) = iq^2 G_M^2,$$

Polarized Lepton tensor

- for longitudinally polarized electrons, one defines the helicity $\lambda = \pm 1$ ($\rightarrow$ spin parallel or antiparallel to the electron three-momentum).

The general expression for the leptonic tensor is :

$$L_{\mu\nu} = L^{(0)}_{\mu\nu} + L_{\mu\nu}(\lambda_1) + L_{\mu\nu}(\lambda_2) + L_{\mu\nu}(\lambda_1, \lambda_2).$$

Initial electron polarized : $\lambda_1 = \lambda$:

$$L^{(1)}_{\mu\nu} = \frac{1}{2} Tr\gamma_\mu \hat{k}_1 \gamma_\nu \hat{k}_2 \gamma_5 = -\frac{1}{2} Tr\gamma_\mu \gamma_\nu \hat{k}_1 \hat{k}_2 \gamma_5 = 2i\epsilon_{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma} \qquad (2)$$

We use the property of γ_5 : $Tr\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 = -4i\epsilon_{\mu\nu\rho\sigma}$.

$$L_{\mu\nu}(\lambda) = 2i\lambda\epsilon_{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}.$$

Polarized electron: antisymmetric tensor

If the initial proton is unpolarized (symmetric tensor): no single spin polarization (as long as FFs are real) *Only double spin observables in ep scattering*

Recoil Proton Polarization (Px,Pz)

$$\begin{aligned} L_{\mu\nu}(\lambda)W_{\mu\nu}(P_x) &= L_{0y}(\lambda)W_{0y}(P_x) + L_{y0}(\lambda)W_{y0}(P_x) \\ &= L_{0y}(\lambda)\left[W_{0y}(P_x) - W_{y0}(P_x)\right] \\ &= 2L_{0y}(\lambda)W_{0y}(P_x). \end{aligned}$$

$L_{0y} = 2i\lambda\epsilon_{0y\alpha\beta}k_{1\alpha}k_{2\beta} \rightarrow$ only non-zero terms for $\alpha = x$ and $\beta = z$ or $\alpha = z$ and $\beta = x$.

$$\begin{aligned} L_{0y}(\lambda) &= 2i\lambda\left(\epsilon_{0yxz}k_{1x}k_{2z} + \epsilon_{0yzx}k_{1z}k_{2x}\right) \\ &= 2i\lambda\epsilon_{0yxz}(k_{1x}k_{2z} - k_{1z}k_{2x}) = i\lambda q^2 \cot\frac{\theta_B}{2}, \end{aligned}$$

with $\epsilon_{0yxz} = 1$.

$$L_{\mu\nu}(\lambda)W_{\mu\nu}(P_x) = -4\lambda m q^2\sqrt{-q^2}\cot\frac{\theta_B}{2}G_E G_M.$$

Px

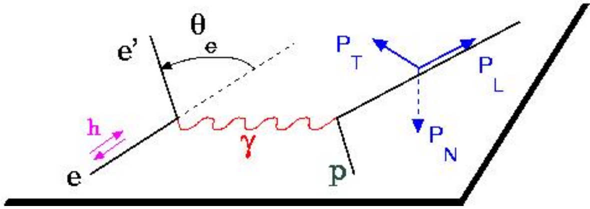
$$\begin{aligned} L_{\mu\nu}(\lambda)W_{\mu\nu}(P_z) = \; & 2i\lambda\epsilon_{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}W_{\mu\nu}(P_z) = \\ & 4\epsilon_{xy0z}W_{xy}(P_z)\left(\epsilon_{1B}k_{2B}^z - \epsilon_{2B}k_{1B}^z\right) = \\ & 4\lambda q^2\frac{G_M^2}{\sin\theta_B/2}. \end{aligned}$$

Pz

Final formulas

The polarization **P** of the scattered proton can be written as :

$$\mathbf{P}\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{4\pi^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \frac{L_{\mu\nu}}{m^2}\vec{P}_{\mu\nu}.$$



with $\vec{P}_{\mu\nu} = \frac{1}{2}(\mathcal{T}r\mathcal{F}_\mu\mathcal{F}_\nu^\dagger\vec{\sigma})$, so that $P_{\mu\nu}^{(z)} = W_{\mu\nu}(P_z)$ and $P_{\mu\nu}^{(x)} = W_{\mu\nu}(P_x)$

The components P_x and P_z of the proton polarization vector (in the scattering plane) are

$$DP_x = -2\lambda\cot\frac{\theta_e}{2}\sqrt{\frac{\tau}{1+\tau}}G_E G_M,$$

$$DP_z = \lambda\frac{\epsilon_1+\epsilon_2}{m}\sqrt{\frac{\tau}{1+\tau}}G_M^2, \quad D = 2\tau G_M^2 + \cot^2\frac{\theta_e}{2}\frac{G_E^2+\tau G_M^2}{1+\tau}.$$

$$\frac{P_x}{P_z} = \frac{P_t}{P_\ell} = -2\cot\frac{\theta_e}{2}\frac{m}{\epsilon_1+\epsilon_2}\frac{G_E(q^2)}{G_M(q^2)}$$

Recoil proton polarization in ep elastic scattering with *longitudinally polarized electron beam*

Polarized target

$$\frac{d\sigma}{d\Omega_e}(\mathcal{P}) = \left(\frac{d\sigma}{d\Omega_e}\right)_0 (1 + \lambda \mathcal{P}_x A_x + \lambda \mathcal{P}_z A_z)\,,$$

$\mathcal{P}$: polarization vector of the target

Analyzing power: same value as the recoil proton polarization P but opposite sign

$$\begin{aligned} A_x &= P_x\,, \\ A_z &= -P_z\,. \end{aligned}$$

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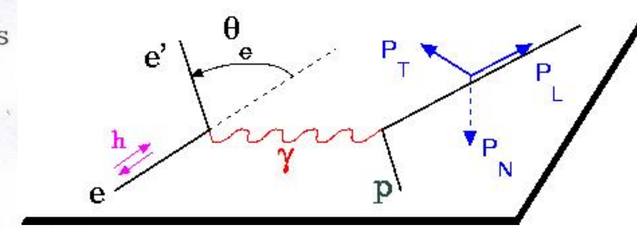

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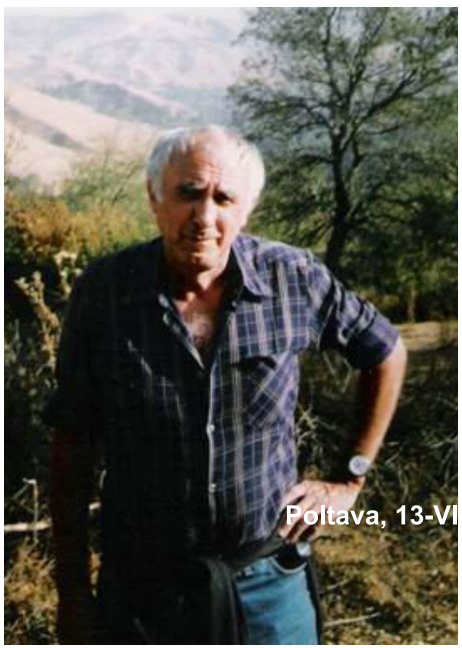
M.P. Rekalo (1938-2004)



$$s_2 \frac{d\sigma}{d\Omega_R} = 4p_2 \frac{(s\cdot q)}{1+\tau} \Gamma(\theta, \varepsilon_1) \Big[\tau G_M (G_M + G_E) - \frac{1}{4\varepsilon_1} G_M (G_E - \tau G_M) \Big],$$

A.I. Akhiezer (1911-2000)

The polarization induces a term in the cross section proportional to $G_E\, G_M$

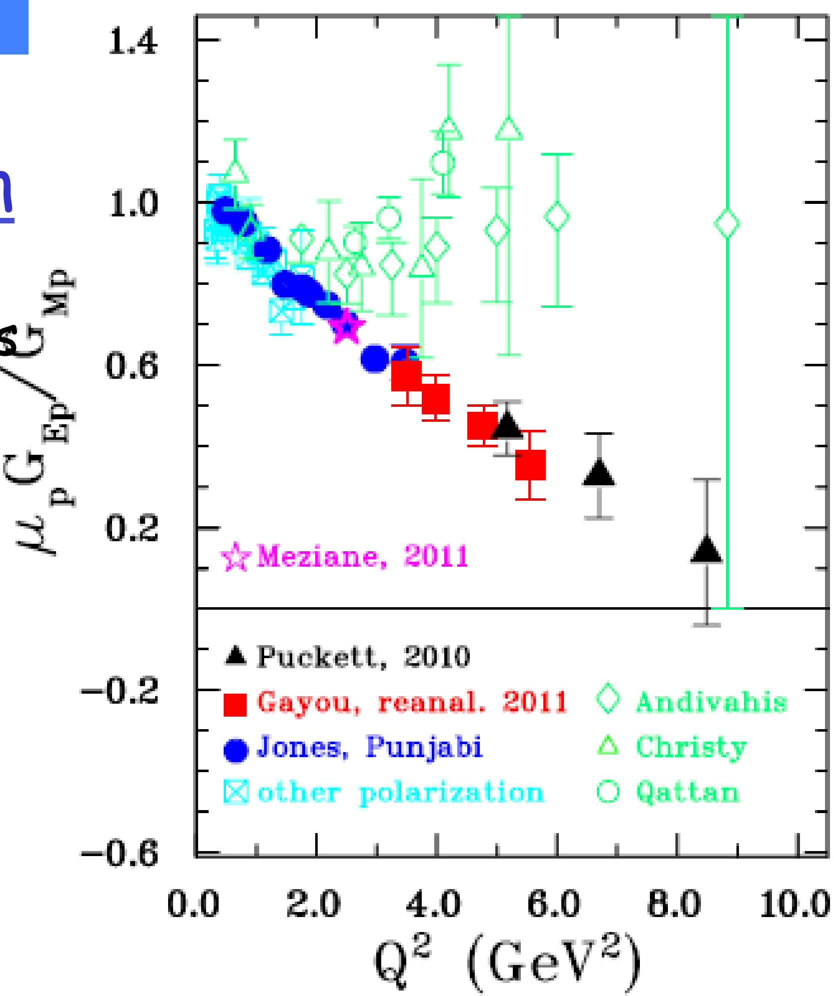
Polarized beam and target or polarized beam and recoil proton polarization

Polarization experiments

A.I. Akhiezer and M.P. Rekalo, 1967

Jlab-GEp collaboration

1) "standard" dipole function for the nucleon magnetic FFs GMp and GMn
2) linear deviation from the dipole function for the electric proton FF Gep
3) QCD scaling not reached
3) Zero crossing of Gep?
4) ***contradiction between polarized and unpolarized measurements***



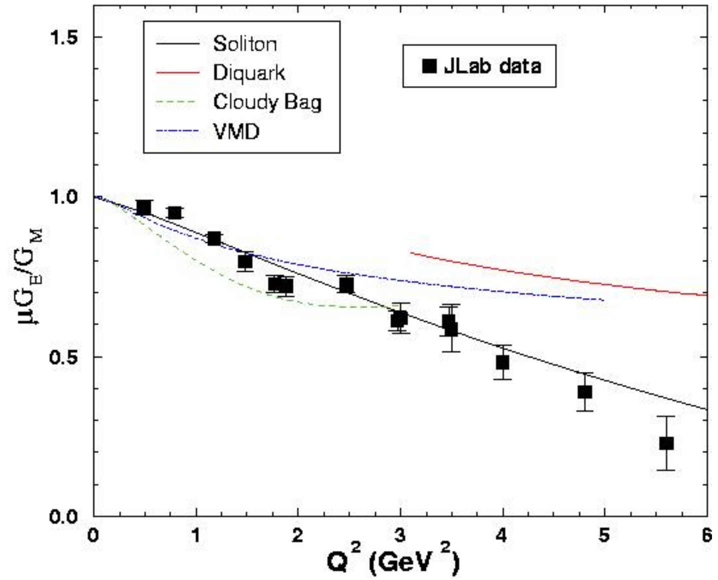
A.J.R. Puckett et al, PRL (2010), PRC (2012)

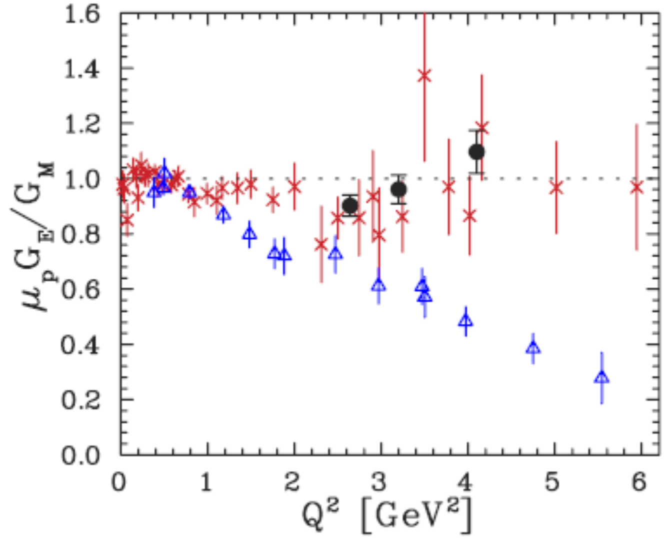
Issues

- Some models (IJL 73, Di-quark, soliton..) predicted such behavior before the data appeared

BUT

- Simultaneous description of the four nucleon form factors...
- ...in the space-like and in the time-like regions
- Consequences for the light ions description
- When pQCD starts to apply?
- Source of the discrepancy





The experimental tools

W. Pauli, N. Bohr

- Accelerators or colliders
- Large acceptance spectrometers and 4π Detectors

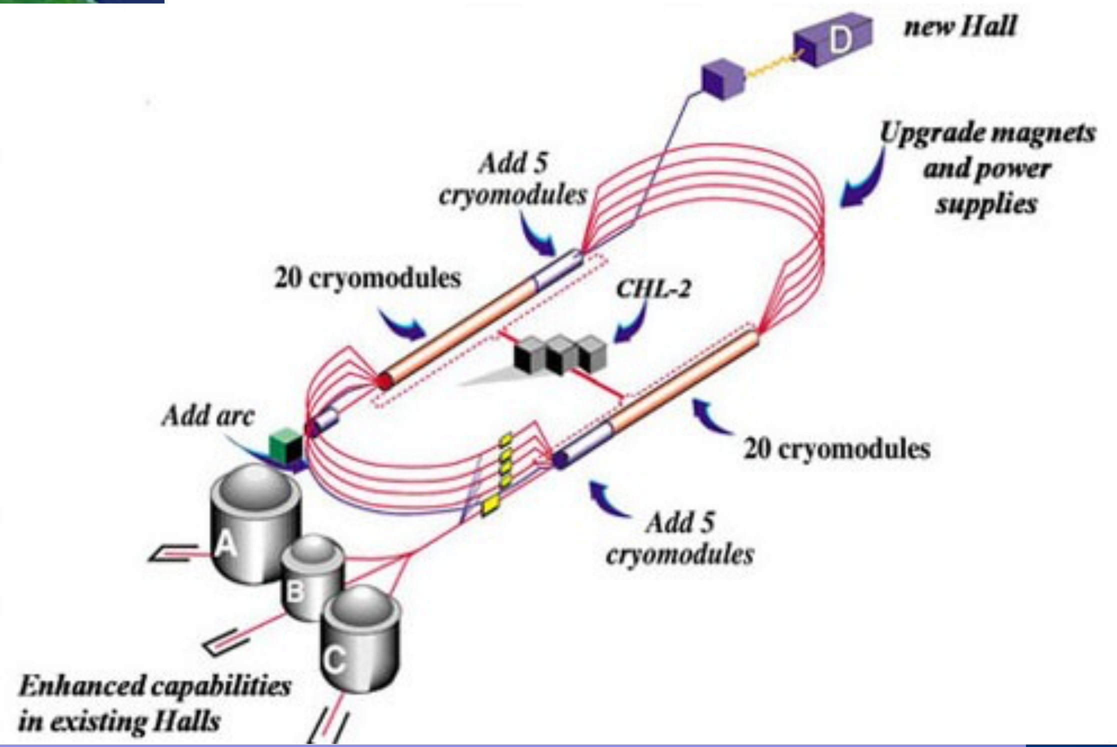
Polarization

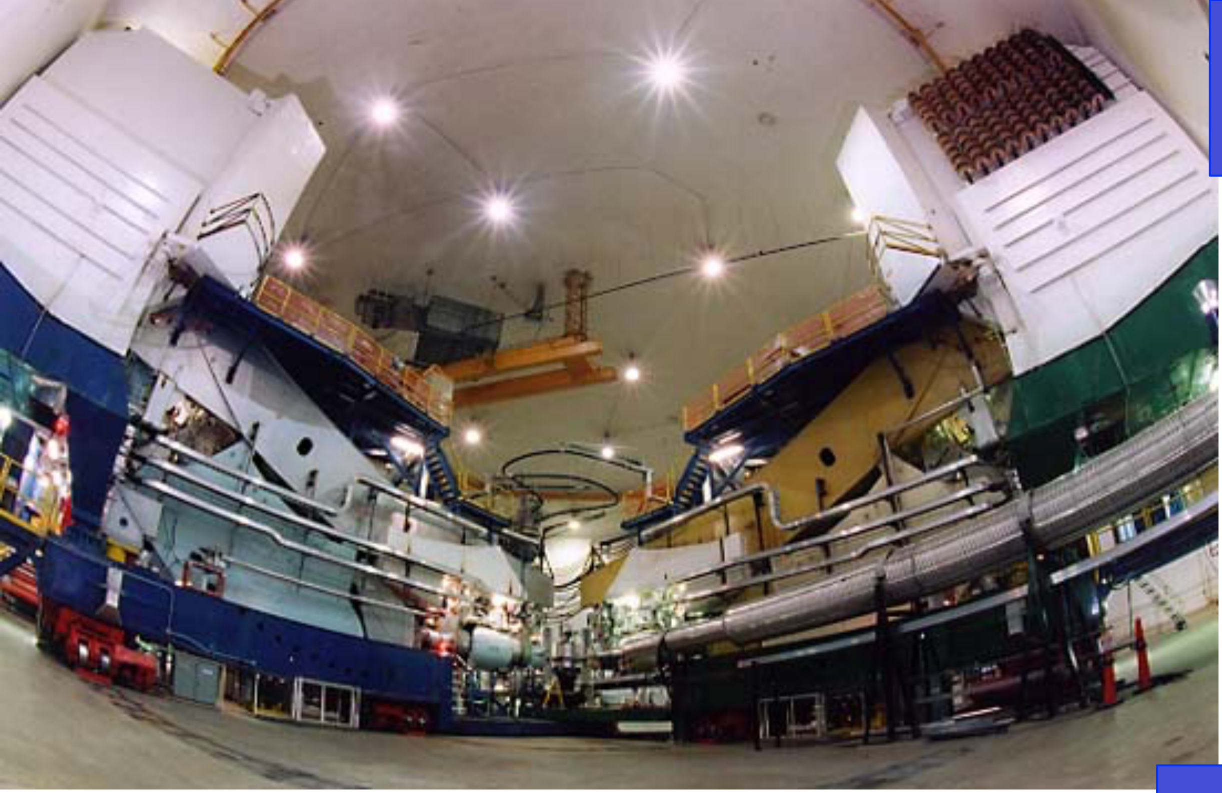
- Polarized beams
- Polarized targets
- Polarimeters
  - Beam polarimeters
  - Polarized proton polarimeters

At each energy (experiment) its own polarimeter
- Polarization experiments are difficult ant time consuming.
-The same physical information with polarized target
or polarimeter ..but polarimeters are in general superior

Jefferson Lab

Virginia, Newport News,Norfolk
1.5 Km long
4, 6 GeV *electron beam*
Highly polarized
0.1 mm e^- *beam size*
Upgraded to 11 GeV



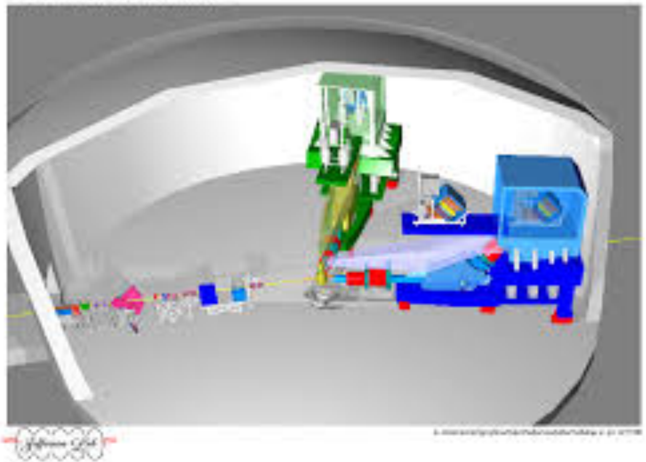


Hall A Jlab
Spectrometer HMS

Characteristics

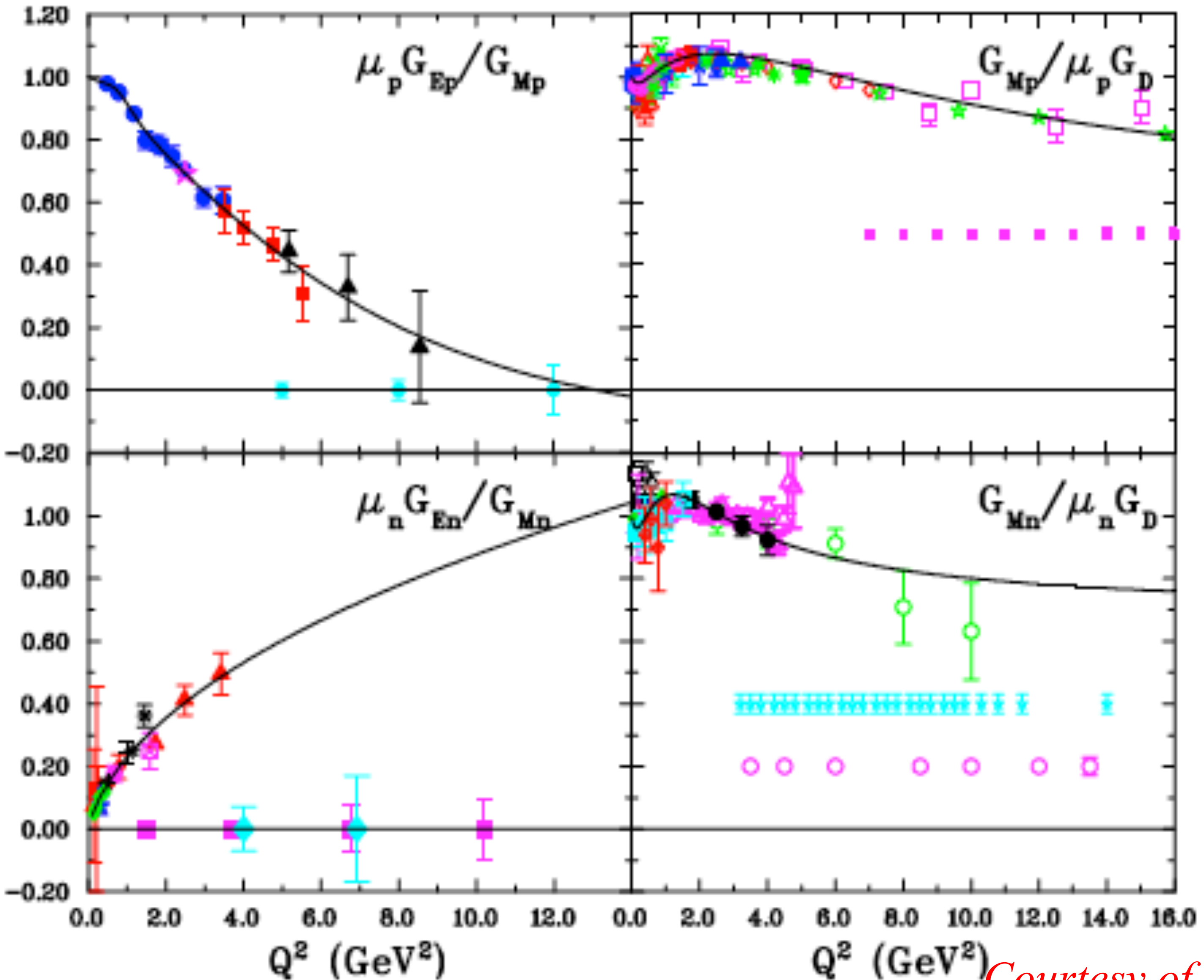
- Momentum [GeV/c] 4
- Acceptance [%] 10
- Resolution $\Delta p/p$ 10^{-4}
- Angular range [deg] 12.5-165
- Solid angle [msr] 7.8

View of Hall A Machines



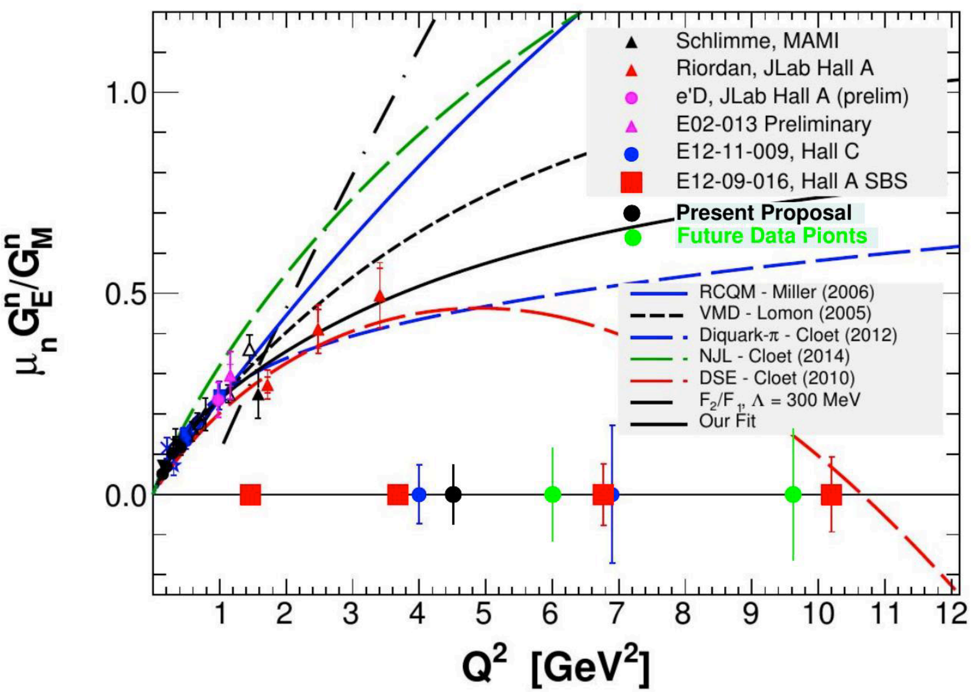
Hall A Jlab
Focal plane polarimeter

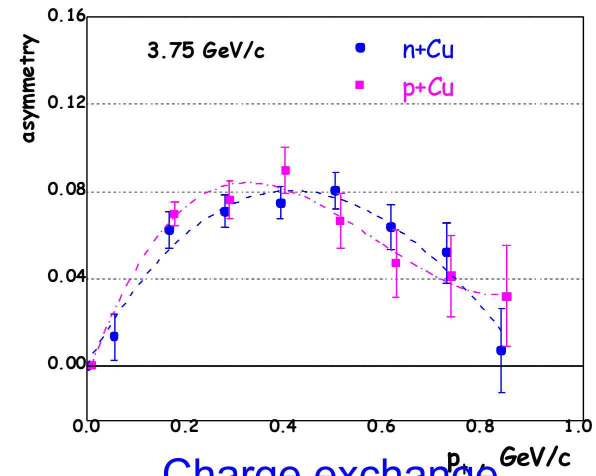
Outlook - Experiments in Halls A, B, and C at 12 GeV



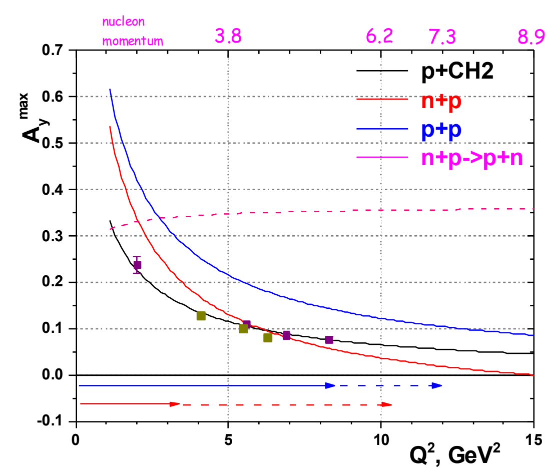
Courtesy of E.J Brash

Neutron Form factors





Charge exchange
Cu target?
Dubna Nuclotron



E12-07-108 GMp elastic p(e,e')p cross section (2%) using High Resolution Spectrometer, max Q2 = 16 (GeV/c)2
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E12-17-004 GEn/GMn (with polarized beam & recoil polarimetry)

Spin & Polarization (I)

- Spin is a fundamental property that characterizes a particles (like mass, charge..)
- A particle of spin S has 2S+1 quantified values of the spin projection:
  a proton: S=1/2, two spin states: ±1/2 up (↑) *and* down (↓)
- A proton beam is not polarized when the two spin directions are equally probable
- The vector polarization is defined as *the difference of population of states up and down* – normalized to the sum

$$P_y = \frac{N(+1/2) - N(-1/2)}{N(+1/2) + N(-1/2)}$$

Deuteron (S=1) has three quantified values of the spin +1,0,-1

Vector polarization:

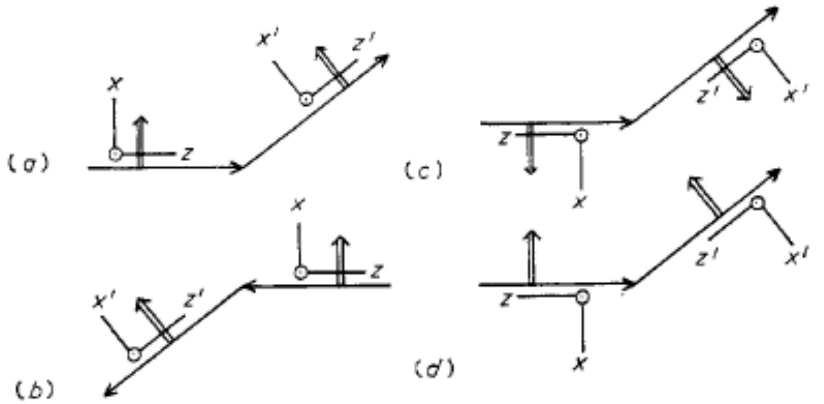
$$P_V = \frac{N(+1) - N(-1)}{N(+1) + N(-1) + N(0)}$$

Tensor polarization

$$P_T = \frac{N(+1) - 2N(0) + N(-1)}{N(+1) + N(-1) + N(0)}$$

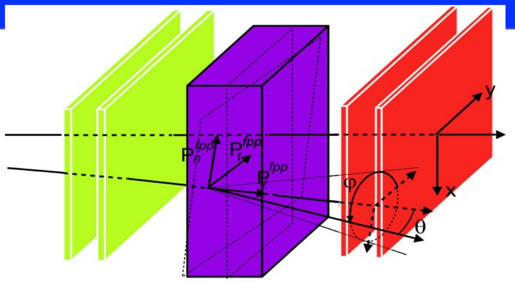
Spin & Polarization (II)

- The wave function of a particle with 0 spin, as α, π, *is described by a (pseudo)scalar*.
- The wave function of a particle with spin 1/2, as a proton, *is described by a two-component spinor*.
- The wave function of a particle with spin 1, as a deuteron, *is described by a three-component vector*.
- The density matrix is an average on an ensemble of particles, quadratic in the wave functions.
  - If a beam is *unpolarized* it is a diagonal, unit, matrix: *all projections are equiprobable*.
- Symmetries apply!
- Parity and time invariance
  - Polar vectors change sign
  - Spin operators DO NOT(pseudovector)



Hadron Polarimetry

Working principle: measurement of
the azymuthal asymmetry in a secondary scattering



Choice of the secondary reaction (1 charged particle+X)

large cross section (statistical errors)

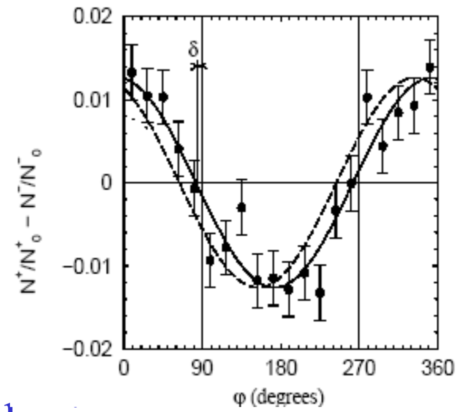
large analyzing power (systematic errors)

Precision on the track reconstruction

Detector alignment:

1 mm by laser

1/10 mm under beam, with particles that do not interact



$$N^{\pm}(\theta,\phi) = N_0(\theta)(1 \pm P_y A_y(\theta)\cos\phi),$$

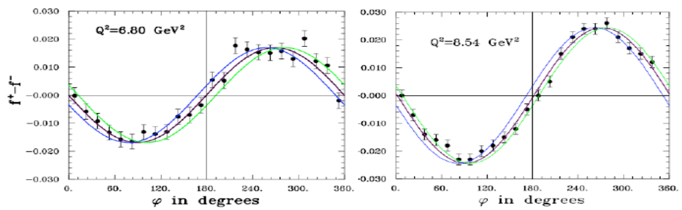
$$R(\theta,\phi) = \frac{N^{+}(\theta,\phi) - N^{-}(\theta,\phi)}{N^{+}(\theta,\phi) + N^{-}(\theta,\phi)} = a_1(\theta)\cos\phi$$

$$A_y(\theta) = \frac{a_1(\theta)}{P_y},\quad \Delta A_y \simeq \frac{1}{P_y}\sqrt{\frac{1}{N_{Incident}}}.$$

1)Calibration : Analyzing powers

2) Measurement : Polarization

Physical Asymmetries at Q^2 of 6.8 and 8.5 GeV2



Hadron Polarimeter

- The efficiency

$$\epsilon(\theta) = \frac{N_{useful}(\theta)}{N_{incident}(\theta)}$$

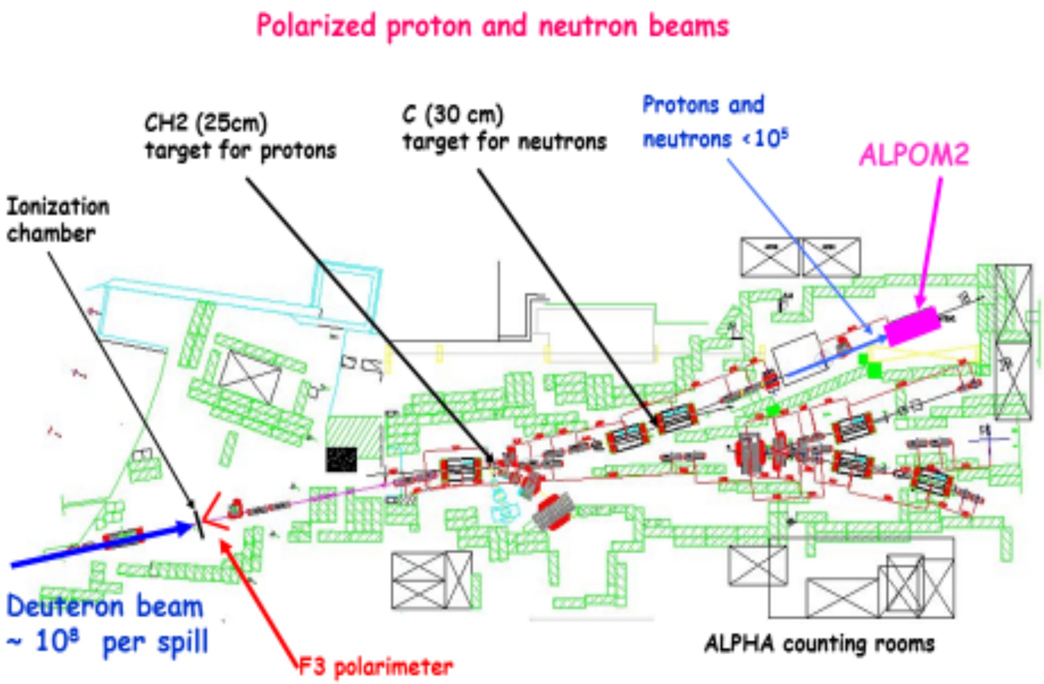
- The figure of merit:

$$\mathcal{F}^2 = \sum_{\theta} \epsilon(\theta)\ \mathcal{A}^2(\theta)$$

- The error on the polarization measurement

$$\Delta P = \sqrt{\frac{2}{N_{incident}(\theta)\mathcal{F}^2}}$$

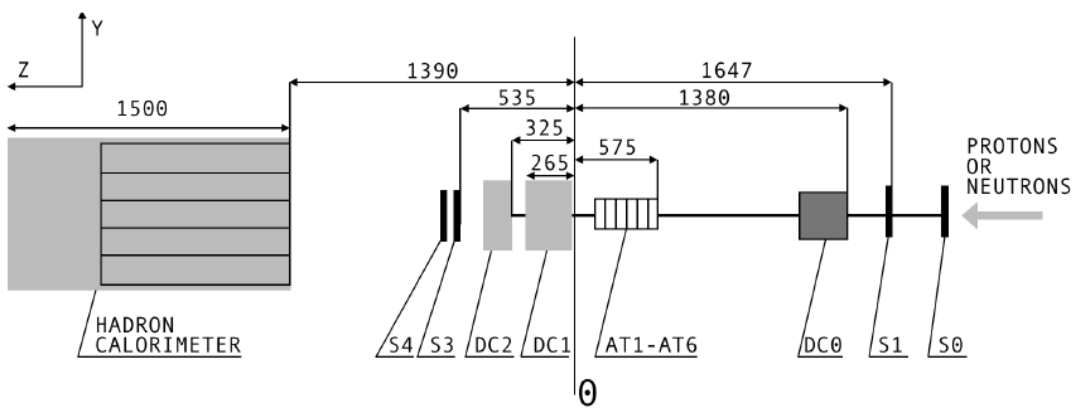
The ALPOM2 Experiment

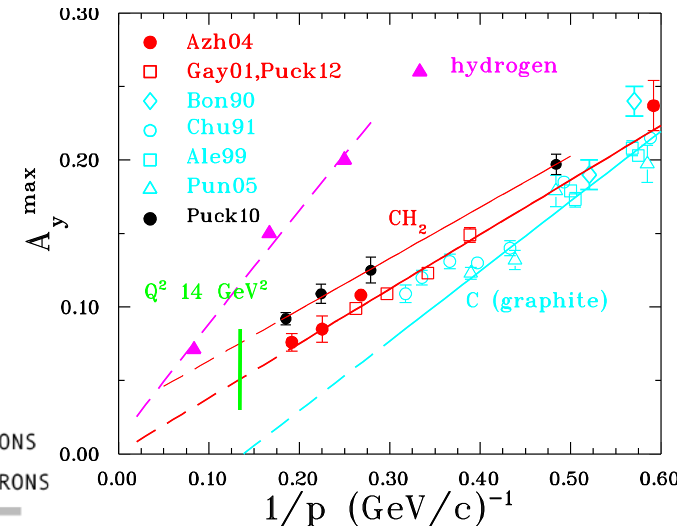


Only in Dubna Nuclotron
13 GeV/c (pol)d !!

(pol)n and (pol) p by break-up
Two major advances:

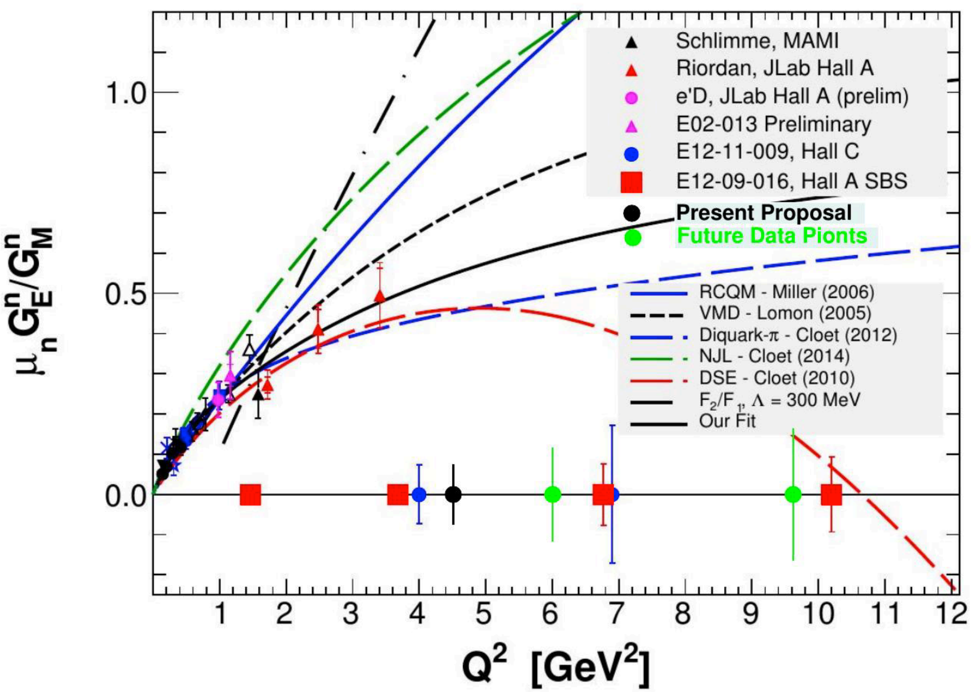
- Charge exchange reaction
- Calorimetry

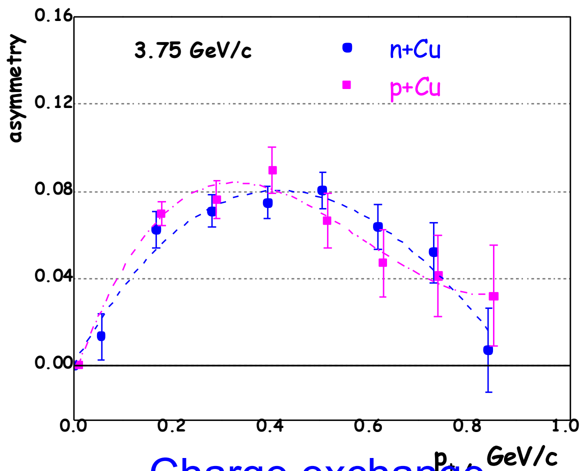




Feasibility studies
For JLab

Neutron Form factors





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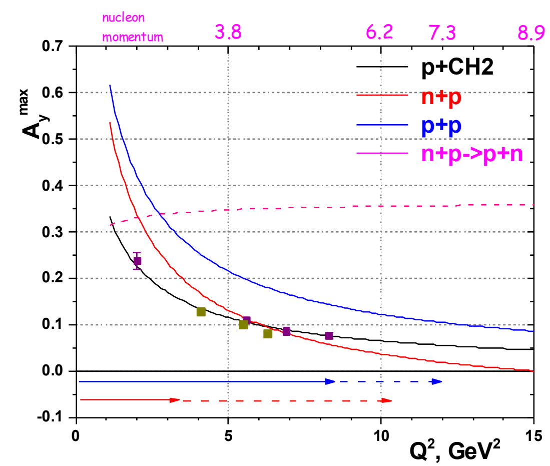

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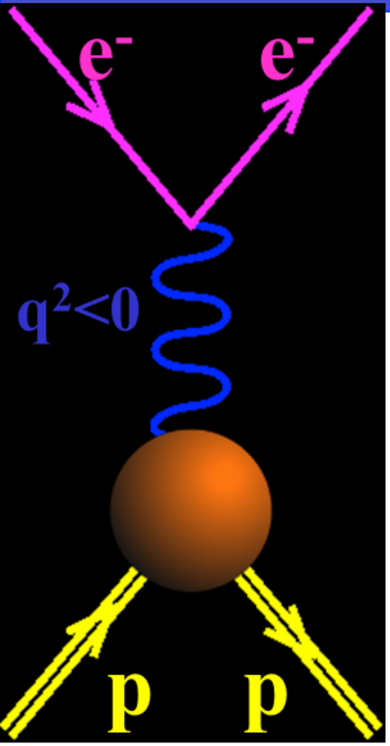
Current conservation



- The EM hadron current

$$\mathcal{J}_\mu = \overline{u}(p_2)\left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p_1),$$
$$\sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2}.$$

Gauge invariance : $\mathcal{J}\cdot q = 0$, for any values of F_1 and F_2, i.e. the current $\mathcal{J}_\mu$ is conserved.

- the term $\sigma_{\mu\nu}q_\mu q_\nu$ vanishes, because it is the product of a symmetrical and antisymmetrical tensors,
- From the Dirac equation : $\overline{u}(p_2)\hat{q}u(p_1) = \overline{u}(p_2)(\hat{p}_2 - \hat{p}_1)u(p_1) = \overline{u}(p_2)(m-m)u(p_1) = 0$.

Holds when nucleons (in initial and final states) are real, F_1 violates the current conservation if one nucleon is virtual.

$$(\hat{k}-m)u(k) = 0, \quad \hat{k} = k\gamma_\mu = E\gamma_0 - \mathbf{k}\cdot\vec{\gamma},$$

The Dirac Equation

$$(\hat{k} - m)u(k) = 0, \quad \hat{k} = k\gamma_\mu = E\gamma_0 - \mathbf{k}\cdot\vec{\gamma},$$

The relativistic description of the spin properties of the particles is based on the Dirac equation.

particle $\quad$ *antiparticle*

$$(\hat{k} - m)u(k) = 0 \qquad (\hat{k} + m)v(k) = 0$$

$$u(k) = \sqrt{E+m}\begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot\mathbf{k}}{E+m}\chi \end{pmatrix} \quad v(k) = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\mathbf{k}}{E+m}\chi \\ \chi \end{pmatrix}$$

- $k = (E, \mathbf{k})$ is the particle four momentum
- $u(k)$ is a four-component Dirac spinor
- χ is a two-component spinor
- Relativistic invariant normalization : $u^\dagger u = 2E$.

Changing only $E \to -E$ and $k \to -k$ only would be a bad representation as it would create a pole for on mass shell particles.

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

Dirac matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

The density matrix

The density matrixes for polarized and unpolarized particles and antiparticles : $\rho_{\alpha\beta} = u_\alpha(p) u^\dagger_\beta(p)$:

	particle	antiparticle
unpolarized	$\hat{p} + m$	$\hat{p} - m$
polarized	$(\hat{p} + m)\frac{1}{2}(1 - \gamma_5 \hat{s})$	$(\hat{p} - m)\frac{1}{2}(1 - \gamma_5 \hat{s})$

where s_α is the four vector of the electron spin.

$$s_\alpha = \left(\mathbf{s} + \frac{(\mathbf{s}\cdot\mathbf{p})\mathbf{p}}{m_e(\epsilon + m_e)}, \frac{\mathbf{s}\cdot\mathbf{p}}{m_e} \right) \quad s \cdot p = 0, \quad s^2 = -1$$

For relativistic electrons, $\epsilon \gg m_e \quad p_\alpha = \epsilon(\mathbf{1}, 1)$,

$$s_\alpha = \frac{\epsilon}{m} s_\ell(\mathbf{1}, 1)$$

-**1** : unit vector along **p**

-$s_\ell = \mathbf{s} \cdot \mathbf{p}/|\mathbf{p}| \equiv \lambda$.

$$s_\alpha = \frac{p_{1\alpha}}{m}\lambda$$

The density matrix

The density matrices $\rho = u(p)\bar{u}(p)$ for relativistic electrons

$$
\begin{aligned}
\rho &= \frac{1}{2}(\hat{p} + m_e)\left(1 - \gamma_5 \frac{\hat{p}}{m_e}\lambda\right) = \\
&= \frac{1}{2}(\hat{p} + m_e) + \frac{\lambda}{2}(\hat{p} + m_e)\frac{\hat{p}}{m_e}\gamma_5 \\
&= \frac{1}{2}(\hat{p} + m_e) + \frac{\lambda}{2}\left(p^2 + m_e\hat{p}\right)\frac{1}{m_e}\gamma_5 \\
&= \frac{1}{2}(\hat{p} + m_e)(1 + \lambda\gamma_5) \equiv \frac{1}{2}\hat{p}(1 + \lambda\gamma_5),
\end{aligned}
$$

where we used the following property of the γ_5-matrix :
$\hat{p}\gamma_5 + \gamma_5\hat{p} = 0$, for any four-vector p_α.